M.A./M.Sc. Examination 2018

Semester - I

Mathematics

Course: MMC-14 (New)

(Ordinary Differential Equations)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions.

1. a) Use Picard's method of successive approximations and find the first and second approximate solutions of
$$\frac{dy}{dx}=z;\ \frac{dz}{dx}=x^3\left(y+z\right),\ \text{where, } y=1; z=\tfrac{1}{2} \text{ when } x=0. \qquad 4$$
b) State the existence and uniqueness theorem of first order initial value problem. Hence determine the interval of existence of the $IVP: \frac{dy}{dx}=y^2,\ y\left(1\right)=-1,$ where the rectangle R is given by 1+3
$$R:\left\{\left(x,y\right):\left|x-1\right|\le a,\ \left|y+1\right|\le b\right\}$$
c) Consider $f\left(x,y\right)=x^3\left|y\right|$. Prove that f satisfies a Lipschitz condition on $R:\left|x\right|\le 2,\ \left|y\right|\le 2$ but $\frac{\partial f}{\partial y}$ does not exist at $\left(x,0\right)$ if $x\ne 0$. 2

2. a) What is a fundamental matrix of vector differential equation? Show that if $\phi\left(t\right)$ is a fundamental matrix of $\frac{dx}{dt}=A\left(t\right)X,$ then the solution of $\frac{dx}{dt}=A\left(t\right)X+F\left(t\right)$ can be assumed as $X\left(t\right)=\phi\left(t\right)\int_{t_0}^{t}\phi^{-1}\left(u\right)F\left(u\right)du.$ 4
b) Find the unique solution of the differential equation $\frac{dx}{dt}=\begin{pmatrix}6 & -3\\ 2 & 1\end{pmatrix}X+\begin{pmatrix}e^{5t}\\ 4\end{pmatrix}$ that satisfies the initial condition $\phi\left(0\right)=\begin{pmatrix}9\\ 4\end{pmatrix}$. 4
c) Is $\phi\left(t\right)=\begin{pmatrix}2e^{t} & -e^{-3t}\\ -4e^{t} & 2e^{-3t}\end{pmatrix}$ a fundamental matrix for the system $\dot{x}=Ax$? 2

3. a) If $P^{-1}AP=D,\ D$ being a diagonal matrix, then show that $e^{At}=Pe^{Dt}P^{-1}=P\begin{bmatrix}e^{\lambda_1 t} & 0\\ - & e^{\lambda_2 t}\end{bmatrix}P^{-1},$ where $D=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$. 3
b) Show that, $e^{A+B}=e^{A}e^{B}$, provided $AB=BA$. 3

P.T.O.

c) Write down the theorem for solving the homogeneous system of first order ODE,

$$\frac{dx}{dt} = AX,$$ 4

where the matrix A has repeated eigen values.

Hence solve

$$\dot{x} = 5x + 4y$$

$$\dot{y} = -x + y$$

4. a) Find all the eigen values and eigen functions of the sturm-Liouville problem, $y'' + \lambda y = 0,$ with $y(0) + y'(0) = 0$ and $y(1) + y'(1) = 0.$ 5

b) Find the Green's function of the ODE:

$\frac{d^2y}{dx^2} - y = 0$ with $y(0) = 0 = y(1)$. Hence solve $\frac{d^2y}{dx^2} - y = x.$ 5

5. a) If f be the solution of $\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + \varphi(t)x = 0$ having first derivative f' on $a \le t \le b$ and has a infinite number of zeros on $a \le t \le b,$ then show that $f(t) = 0$ for all t on $a \le t \le b.$ 3

b) If f and g be any two solution of

$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + \varphi(t)x = 0$ on $a \le t \le b,$ then show that

$P(t)\left[f(t)g'(t) - f'(t)g(t)\right]$ is constant for all $t \in [a,b].$ 3

c) If f and g be the two solutions of

$$\frac{d}{dt}\left[P(t)\frac{dx}{dt}\right] + \varphi(t)x = 0$$

Such that f and g have a common zero, then show that f and g are linearly independent on $a \le t \le b.$ 4

6. a) Find the condition under which the fixed point of the system:

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$$ is a centre. 3

b) Show that the linearized system corresponding to

$$\dot{x} = -y + px\left(x^2 + y^2\right),$$

$$\dot{y} = x + py\left(x^2 + y^2\right)$$ 4

(P is a parameter), predicts the fixed point as a centre $\forall p.$. Show further that the fixed point is a stable spiral if $p < 0$ and unstable spiral if $p > 0$ for the original system.

c) Linearize the system $\dot{x} = e^{-x-3y} - 1,\ \dot{y} = -x\left(1 - y^2\right)$ about the fixed point (0, 0). Also classify the fixed point. 2+1