M.A./M.Sc. Examination 2018

Semester - I

Mathematics

Course: MMC-13 (Old)

(Algebra-I)

(For Back Candidates)

Time: Three Hours | Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four***.

1. a) Let G_1 and G_2 be two groups and $a \in G_1, b \in G_2$. Show that $0((a,b)) = \text{lcm}(0(a), 0(b))$. Also find the number of generators of $\mathbb{Z}_{29} \times \mathbb{Z}_{37}$. 3

   b) Show that the group $(\mathbb{Z}, +)$ can not be expressed as an internal direct product of two nontrivial subgroups. 2

   c) Find the conjugacy class equation of S_3. 2

   d) Let H be a subgroup of a group G. Show that H is a normal subgroup of G if and only if H is a union of conjugacy classes of G. 3

2. a) State and prove Cauchy's theorem for abelian groups. 4

   b) Let G be a noncyclic group of order 21. Find the number of elements of order 3 in G. 3

   c) Let G be a nontrivial finite group. Show that G is a p-group if and only if $|G| = p^n$ for some $n > 0$. 3

3. a) State and prove Sylow's first theorem. 4

   b) Show that every group of order 99 is abelian. 3

   c) Let G be a group. Prove that G is solvable if and only if $G^{(n)} = \{e\}$ for some positive integer n. Hence or otherwise show that S_5 is not solvable. 3

4. a) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

   $$T(x_1 y, z) = (x + y - 2z, \ -x + 2y - z, \ 2x + 5y - 5z).$$

   Check the rank-nullity theorem for T. 4

   b) Let $T : V \to V$ be a linear transformations. If the matrix representations of T relative to any basis are the same, then show that $T = cI_V$, for some $c \in F$. Does the converse hold? Justify your answer. 3

   c) Find the change of basis matrix P from the basis $\beta = \{(1,0,0), (1,2,1), (1,3,1)\}$ into the basis $\beta' = \{(1,0,0), (1,1,0), (1,1,1)\}$. If Q is the change of basis matrix from β' into β, write the relation of P and Q. 3

P.T.O.

5. a) Let V be a vector space having a basis $\beta=\{e_1, e_2, \ldots, e_n\}$. Find a basis of V^* and show that dim $V^*=n$. 3

b) Let V be a finite dimensional vector space of dimension n. Show that $f, g \in V^*$ are linearly independent if and only if $\dim(\ker f \cap \ker g)=n-2$. 4

c) If $U=L(\{(1,2,1),(2,-3,1)\})$, then find U^0. 3

6. a) Find all eigen values and a basis of the eigen space of the eigen value of algebraic multiplicity 2, of T, where $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T(x, y, z)=(2x+y,\ y-z,\ 2y+4z)$. 4

b) Let $T: V \to V$ be a linear transformation and W be a T-invariant subspace of V Prove that $\chi_{T_W}(t) \mid \chi_T(t)$. 3

c) State and prove the Cayley-Hamilton theorem for linear transformations. 3