M.A./M.Sc. Examination 2018

Semester - I

Mathematics

Course: MMC-13 (New)

(Algebra-I)

Time: Three Hours — Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions.

1. a) Let G_1 and G_2 be two groups and $a \in G_1, b \in G_2$. Show that $0((a,b)) = \text{lcm}(0(a), 0(b))$. Also find the number of generators of $\mathbb{Z}_{30} \times \mathbb{Z}_{77}$. 3

   b) Show that the group $(\mathbb{Q}, +)$ can not be expressed as an internal direct product of two nontrivial subgroups. 2

   c) Find the conjugacy class equation of S_4. 2

   d) Let H be a subgroup of a group G. Show that H is a normal subgroup if and only if H is a union of conjugacy classes of G. 3

2. a) State and prove Cauchy's theorem for abelian groups. 4

   b) Let G be a noncyclic group of order 21. Find the number of elements of order 3 in G. 3

   c) Let G be a finite group and p be the smallest prime divisor of $|G|$. Then prove that every subgroup H of G of index p is normal in G. 3

3. a) Let G be a group of order $p^n m$ where p is a prime and $n \in \mathbb{N}$. If H is a subgroup of G of order p^i, $i < n$ then show that G has a subgroup K of order p^{i+1} such that $H \subseteq K$. 2

   b) Show that every group of order 99 is abelian. 3

   c) Let H and K be two subgroups of a group G such that H is normal in G. If both H and K are solvable then show that HK is solvable. 2

   d) Let G be a group. Prove that G is solvable if and only if $G^{(n)} = \{e\}$ for some positive integer n. Hence or otherwise show that S_5 is not solvable. 3

4. a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(x, y) = (x + y,\ x - y)$ for all $(x, y) \in \mathbb{R}^2$. Find the image of the line $2x + 3y = 6$ under T. 2

   b) Find the rank of the linear transformation $T : P_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(p(t)) = \begin{pmatrix} f(0) + f(1) & 0 \\ 0 & f(2) \end{pmatrix}$. 2

P.T.O.

c) Let $T:V\rightarrow V$ be a linear transformation. If the matrix representations of T relative to any basis are the same then show that $T=c\,I_V$. For some $c\in F$. Does the converse hold? 3

c) Let P be the change of basis matrix from a basis β into the basis β'. Show that P is invertible and P^{-1} is the change of basis matrix from β' into β. 3

5. a) With the help of a suitably defined mapping from V onto V^{**}, show that V and V^{**} are isomorphic. 3

b) Let V be a finite dimensional vector space of dimension n. Show that $f,g\in V^*$ are linearly independent if and only if $\dim(\ker f\cap\ker g)=n-2$. What happens, if we consider three linear functionals? 4

c) Let $U=L\left(\left\{(1,2,1),\,(2,-3,1)\right\}\right)$. Find U^0. 3

6. a) Find all eigen values and a basis of the eigen space of the eigen value of algebraic multiplicity 2 of $T:\mathbb{R}^3\rightarrow\mathbb{R}^3$ defined by $T(x,y,z)=(2x+y,\ y-z,\ 2y+4z)$. 4

b) Let $T:V\rightarrow V$ be a linear transformation. If W is a T-invariant subspace of V, prove that $\chi_{T_W}(t)\mid\chi_T(t)$. Hence or otherwise prove that geometric multiplicity of an eigen value cannot exceed its algebraic multiplicity. 3

c) Let $T:V\rightarrow V$ be a linear transformation such that V is T-cyclic. If $S:V\rightarrow V$ is a linear transformation such that $ST=TS$, then show that $S=g(T)$ for some polynomial $g(t)$. 3

———