

# M.A./M.Sc. Examination 2018

Semester – I

Mathematics

Course: MMC-12 (Old)

(Complex Analysis)

(For Back Candidates)

Time: Three Hours

Full Marks: 40

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings.

Answer **any four** questions.

1. a) Evaluate  $\int_{\Gamma} \operatorname{Im} z$ , where  $\Gamma$  is the contour joining  $(0,0)$  and  $(1,1)$  by a straight line. 3

b) If  $f(z)$  be analytic in  $\Omega$  and let  $\Gamma_1, \Gamma_2$  be piecewise non-intersecting smooth curves from the point  $z_1$  to the point  $z_2$ , all lying inside  $\Omega$  and such that all points between the two curves are also in  $\Omega$ . Show that

$$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz. \quad 3$$

c) State Cauchy's Fundamental theorem. Hence evaluate

$$\oint_{|z|=1} \frac{dz}{z+2} \text{ and hence show that } \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0. \quad 1+3$$

2. a) Evaluate  $\oint_{\Gamma} \frac{e^{2z}}{(z+1)^4} dz$ , where  $\Gamma$  is the circle  $|z|=3$ . 2

b) State Liouville's theorem.

If  $f(z)$  is analytic for all finite values of  $z$  and if  $|f(z)| \geq k > 0$ , then prove that  $f(z)$  is a constant. 1+2

c) State and prove Minimum Modulus theorem. 1+4

3. a) If  $|f(z)| > m$  on  $|z|=r$ ,  $f(z)$  is regular for  $|z| \leq r$  and  $|f(0)| < m$ , then show that  $f(z)$  has at least one zero in  $|z| < r$ . 3

b) State and prove Morera's theorem. 1+3

c) State Laurent's theorem.

Expand  $f(z) = \frac{1}{z(z-1)(z-2)}$  in a Laurent series which is valid for  $1 < |z| < 2$ . 1+2

4. a) Define a 'zero' of an analytic function.

If a function  $f(z)$  is regular in a domain  $D$  and if  $z_1, z_2, \dots, z_n, \dots$  is a sequence of zeros of  $f(z)$  having a limit point as an interior point  $\alpha$  of  $D$ , then show that  $f(z) \equiv 0$ . 1+3

P.T.O.

b) Find out the zeros and determine the nature of the singularity of the function  $\frac{z-2}{z^2} \sin\left(\frac{1}{z-1}\right)$  and also find the residue of it at  $z = 0$ . 4

c) If  $f(z)$  has a pole of multiplicity  $m$  at  $z = \alpha$ , then show that  $\text{Res}[f(z); \alpha] = \frac{1}{(m-1)!} \lim_{z \rightarrow \alpha} \frac{d^{m-1}}{dz^{m-1}} [(z - \alpha)^m f(z)]$ . 2

5. a) If  $z = \alpha$  is a zero of  $f(z)$  of order  $m$  then show that  $\frac{1}{f(z)}$  has a pole of order  $m$  at  $z = \alpha$ . 3

b) State and prove Rouché's theorem. (Principle of Argument may be assumed) 4

c) Let  $f(z)$  be analytic in  $|z| \leq 1$  and let  $|f(z)| < 1$  on  $|z| = 1$ . Show that there exists just one point  $\alpha$  within the circle  $|z| = 1$  such that  $f(\alpha) = \alpha$ . 3

6. a) Evaluate the integral

$$\oint_C \frac{f'(z)}{f(z)} dz, \text{ where } f(z) = \frac{z^2 + 1}{(z^2 + 3z + 2)^3} \text{ and}$$

$C$  is the circle  $|z| = 3$ . 2

b) Calculate the residue of  $\frac{z^2}{z^2 - 1}$  at the point at infinity. 3

c) Evaluate by the method of contour integration (any one) of the following: 5

(i)  $\int_0^\infty \sin x^2 dx$

(ii)  $\int_0^\infty \frac{\sin x}{x} dx$ .

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