M.A./M.Sc. Examination 2018

Semester - I

Mathematics

Course: MMC-12 (New)

(Complex Analysis)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions.

1. a) Evaluate $\int_C \bar{z}\, dz$, where C is the curve $z(t) = t^2 + it$ joining the points $z = 0$ and $z = 4 + 2i.$ 2

   b) State Cauchy's Fundamental theorem. 1+2
   Show by an example that the conditions assumed in Cauchy's Fundamental theorem are sufficient but not necessary.

   c) State and prove Cauchy's integral formula. 3

   d) Evaluate $\oint_C \frac{z^2 dz}{(2z-1)(z+2)}$, where C is the circle $|z-1| = 1.$ 2

2. a) The function $f(z)$ is regular in $|z| < R_1$. If $|a| < R < R_1$, then prove that $f(a) = \frac{1}{2\pi i}\oint_C \frac{(R^2 - a\bar{a})}{(z-a)(R^2 - z\bar{a})} f(z)\, dz,$ where C is the circle $|z| = R.$ 2

   b) State and prove Liouville's theorem. 1+2

   c) State and prove fundamental theorem of algebra. 1+2

   d) Show that $f(z) = z^9 + z^4 + 1$ vanishes at least for one point in $|z| < 2.$ 2

3. a) State Talylor's theorem.
   Expand $f(z) = \frac{1}{z-4}$ in its power series about $z = 3.$ 1+2

   b) State and prove Laurent's theorem. 1+4

   c) Prove that $e^{z+\frac{1}{z}} = \sum_{-\infty}^{\infty} a_n z^n$ where $\pi a_n = \int_0^{\pi} e^{2Cos\theta} \cos n\theta d\theta.$ 2

4. a) Prove that a function f has a pole at α of order m if and only if f can be expressed in the form $f(z) = \frac{\psi(z)}{(z-\alpha)^m}$, where $\psi(z)$ is analytic at α and $\psi(\alpha) \neq 0.$ 3

   b) Show that the sum of the residues of $\frac{e^{-z}}{z^2 - a^2}$ at its pole is $-\frac{1}{a}\sinh a.$ 2

   c) If $f(z)$ has a pole at α of order m, then show that $|f(z)| \to \infty$ as $z \to \alpha$ in any manner. 3

   d) Prove that the zeros of a non-constant analytic function are isolated points. 2

P.T.O.

5. a) If $f(z)=\frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are analytic at α and $P(\alpha)\neq 0$, while α is a simple zero of $Q(z)$, then show that α is a simple pole of $f(z)$ with residue $\frac{P(\alpha)}{Q'(\alpha)}$. 2

b) Find the singularities of the function $f(z)=\tan\frac{1}{z}$ and determine the nature of it. 3

c) State and prove Cauchy's residue theorem. 1+2

d) Show that all the roots of the equation $z^5+az+1=0$ lie within the circle $|z|=r$ if $|a|<r^4-\frac{1}{r}$. 2

6. a) Define residue of a function at the point at the infinity. Find $\text{Res}\left[f(z),\infty\right]$, where, $f(z)=\frac{z}{z^2-1}$. 1+2

b) A function which is analytic everywhere including the point at infinity is a constant – prove it. 2

c) Evaluate ***any one*** of the following by the method of contour integration: 5

i) $\int_{-\infty}^{\infty}\frac{dx}{a^2+x^2},\ a<0;$

ii) $\int_{-\infty}^{\infty}\frac{x\sin x}{a^2x^2+b^2}dx,\ a,b>0.$

———