

**M.A./M.Sc. Examination 2018**  
**Semester - III**  
**Mathematics**  
**Optional Course: MMO-31 (P10) (New)**  
**( Rings and Modules-I )**

**Time: Three Hours**

**Full Marks: 40**

Questions are of value as indicated in the margin.  
 Notations and symbols have their usual meanings.  
 Attempt **any four** questions.

1. a) Let  $R$  be a commutative ring with 1. If  $R$  is simple, then show that  $R$  is a field. Also prove that  $M_n(R)$  is simple for every simple ring  $R$ . 4
- b) Let  $I$  be an ideal of a ring  $R$ . Show that  $M_n(R)/M_n(I) \cong M_n(R/I)$ . 3
- c) If  $N$  is a submodule of an  $R$ -module  $M$  and  $A$  is a right ideal of  $R$ , then show that  $(M/N)_A = (N+MA)/N$ . 3
2. a) Define direct product of a family of  $R$ -modules. Prove that any two direct products of a family  $\{M_\alpha\}_\alpha$  of  $R$ -modules are isomorphic. 3
- b) Let  $\{M_\alpha\}_\alpha$  be a family of  $R$ -modules and  $N$  be an  $R$ -module. Show that  $Hom_R(\bigoplus_\alpha M_\alpha, N) \cong \prod_\alpha Hom_R(M_\alpha, N)$  as abelian groups. 4
- c) Give an example with justifications of an  $R$ -module  $M$  such that  $M \cong M \oplus M$ . 3
3. a) Give an example of an  $R$ -module  $M$  which has bases of different cardinalities. 4
- b) Prove that every  $R$ -module  $M$  is isomorphic to a quotient of the free  $R$ -module  $F_X$  for some  $X$ . 2
- c) Prove that every commutative ring has the IBN property. 4
4. a) Let  $M$  be a right  $R$ -module. Show that  $M \otimes_R R \cong M$ . 3
- b) Let  $f_1: M_1 \rightarrow M_2, f_2: M_2 \rightarrow M_3, g_1: N_1 \rightarrow N_2$  and  $g_2: N_2 \rightarrow N_3$  be four  $R$ -linear maps. Prove that  $(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \circ f_1) \otimes (g_2 \circ g_1)$  2
- c) Show that  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ . 2
- d) Let  $d$  be the gcd of  $m$  and  $n$ . Show that  $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d$ . 3
5. a) Define subcategory and additive category. 2
- b) Show that in a category every section is injection and every injection is monic. Give an example of a monic morphism which is not an injection. 4
- c) Prove that every monic morphism is an injection in  $Mod_R$  category. 2
- d) Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a covariant functor and  $T: \mathcal{D} \rightarrow \mathcal{E}$  be a contravariant functor. Determine whether  $TF$  is covariant or contravariant. 2

**P.T.O.**

6. a) Let  $O \rightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \rightarrow 0$  be a short exact sequence. If  $M \simeq M_1 \oplus M_2$  then show that the short exact sequence splits. 3
- b) If  $O \rightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \rightarrow 0$  is a split short exact sequence and  $f'$  and  $g'$  are splitting maps for  $f$  and  $g$ , respective, show that  $O \rightarrow M_2 \xrightarrow{g'} M \xrightarrow{f'} M_1 \rightarrow 0$  is a split short exact sequence. 3
- c) Show that the functors  $Hom_R(X, -): Mod_R \rightarrow Ab$  is left exact. 4
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