M.A./M.Sc.Examination 2018
Semester - III
Mathematics
Course: MMO-31(P-2) (New)
(Advanced Functional Analysis-I)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions.

1. a) Give definition of a topological vector space (tvs) and prove that every tvs is regular. 1+2
   b) Let Xbe a tvs. Then prove the following:
      i) there is a neighbourhood (nbd.) base of θ consisting of balancedneighbourhoods (nbds.). 2
      ii) if X is locally convex then there is a nbd. base of θ consisting of convex and balanced nbds. 3
   c) Let X be a tvs and $A \subset X$. Show that $\overline{A} = \bigcap \{A + V;\ V$ is any nbd. of $\theta\}$. 2

2. a) When is a tvs said to be locally compact? Prove that every locally compact tvs is of finite dimension. 4
   b) Let X be a tvs. If X is metrizable by an invariant metric d then prove the following:
      i) $d(nx, \theta) \le nd(x, \theta)$, $x \in X$ and $n \in \mathbb{N}$.
      ii) if $\{x_n\}$ be a sequence in X such that $x_n \to \theta(\text{as } n \to \infty)$ then there exists a sequence $\{\gamma_n\}$ of scalars such that $\gamma_n \to \infty$ and $\gamma_n x_n \to \theta$ as $n \to \infty$. 4
   c) If X is a locally bounded tvs with Heine-Borel property, then prove that X is finite dimensional. 2

3. a) Let X be a linear space and A be a convex and absorbing set in X containing θ. Then show that
      i) $\mu_A(x+y) \le \mu_A(x) + \mu_A(y)$, for all $x, y \in X$;
      ii) $\mu_A(tx) = t\mu_A(x)$, $\forall t \ge 0$ and for all $x \in X$;
      iii) μ_A is a seminorm, if A is balanced set in X;
      iv) if $B = \{x \in X;\ \mu_A(x) < 1\}$ and $C = \{x \in X;\ \mu_A(x) \le 1\}$ then
         $B \subset A \subset C$ and $\mu_A = \mu_B = \mu_C$. 6
   b) If Y is a subspace of a tvs X and Y is an F-space (in respect of the topology inherited from X), then prove that Y is a closed subspace of X. 4

4. a) State and prove Hahn-Banach theorem for a complex linear space (assuming that the corresponding result holds for a real linear space). 4
   b) If f is a continuous linear functional on a subspace M of a locally convex space X, then show that there exists a $T \in X^*$ such that $T = f$ on M. 3
   c) Let $T:(X, \tau) \to (Y, \tau')$ be a linear mapping where (X, τ) and (Y, τ') are two topological vector spaces. If T is continuous at θ in X then show that T is continuous on X. 3

P.T.O.

5. a) Suppose $\Lambda_1, \Lambda_2, ..., \Lambda_n$ and Λ are linear functionals on a linear space X and $N = \{x \in X : \Lambda_1 x = \Lambda_2 x = ... = \Lambda_n x = 0\}$.
Show that the following statements are equivalent:
   i) There are scalars $\alpha_1, \alpha_2, ..., \alpha_n$ such that
   $$\Lambda = \alpha_1 \Lambda_1 + \alpha_2 \Lambda_2 + ... + \alpha_n \Lambda_n.$$
   ii) There exists a scalar $0 < \gamma < \infty$ such that $|\Lambda x| \le \gamma \max_{1 \le i \le n} |\Lambda_i(x)| \forall x \in X$.
   iii) $\Lambda x = 0 \quad \forall x \in N$. 5

   b) Define a seminorm on a linea space. For any seminorm p on a linear space X, prove that the set $\{x \in X : p(x) \le 1\}$ is convex and balanced. 3

   c) If X is a tvs and A, B are bounded subsets of X then prove that $A \bigcup B$ and $A + B$ are also bounded subsets of X. 2

6. a) Define Weak* topology for a normed linear space. State and prove Banach-Alogulu theorem. 8

   b) If $\tau_1 \subset \tau_2$ are topologies on a set X and if τ_1 is Hausdorff and τ_2 is compact then show that $\tau_1 = \tau_2$. 2