M.A./M.Sc.Examination, 2018
Semester - III
Mathematics
Course: MMO-31(P-1) (New)
(Advanced Complex Analysis-I)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions.

1. a) Let $u(x, y)$ be a real-valued harmonic function in a simply connected domain D. Then show that there is an analytic function f in D such that $u = Re\, f$ which is unique to within an additive arbitrary constant. 3

   b) State Poisson's integral formula. Let f be a function regular in the closed disc $|z| \le R$ and let $v(r, \theta)$ be its imaginary part. If $v(r, \theta) \ge 0$, then prove that

   $$\frac{R-r}{R+r} v(0, 0) \le v(r, \theta) \le \frac{R+r}{R-r} v(0, 0), \text{ where } 0 \le r < R.$$ 1+2

   c) Show that for a fix R and ϕ, Poisson's kernel $\dfrac{R^2 - |z|^2}{\left|Re^{i\phi} - z\right|^2}$ is harmonic in the open disc $|z| < R$. 2

   d) Prove that any harmonic function defined on a domain D has the mean value property in D. 2

2. a) State Dirichlet problem for a disc. Show that it has an unique solution. 1+4

   b) If an entire function $f(z)$ is not a constant, then prove that $M(r) \to \infty$ and $r \to \infty$. 2

   c) If $f(z)$ is analytic for all finite values of z and for all large values of

   $$|z|, |f(z)| < \lambda |z|^k \, (k > 0),$$

   then show that $f(z)$ is a polynomial of degree at most $[k]$. 3

3. a) State and prove Hadamard's three circles theorem. 1+3

   b) Prove that $log\, M(r)$ is a convex function of $log\, r$. 2

   c) Let $f(z)$ be an analytic function in $|z| \le R$ and $M(r), A(r)$ denote respectively the maximum value of $|f(z)|, Re\{f(z)\}$ on $|z| = r$. Then show that for $0 \le r < R$,

   $$M(r) < \frac{R+r}{R-r}\left\{A(R) + |f(0)|\right\}.$$ 4

4. a) If $z_1, z_2, ..., z_n, ...$ be any sequence of numbers whose only limiting point is the point at infinity, then show that it is possible to construct an integral function which vanishes at each of the point z_n and nowhere else. 5

   b) Define order of an entire function. Find the order of e^{z^2+z}. 1+2

   c) If ρ_1 and ρ_2 be the orders of the integral functions $f_1(z)$ and $f_2(z)$ respectively, then show that the order of $f_1(z) \cdot f_2(z)$ is at most $max\{\rho_1, \rho_2\}$. 2

P.T.O.

5. a) If $f(z)$ is an integral function, then the derived function $f'(z)$ is of the same order as $f(z)$ - prove it. 3

b) State and prove Borel's first theorem. 1+4

c) If $f(z)$ is an entire function of finite order ρ and $r_1, r_2, \ldots, r_n, \ldots$ are the moduli of the zeros of $f(z)$ then show that $\sum_{n=1}^{\infty} \frac{1}{r_n^{\alpha}}$ converges if $\alpha > \rho$. 2

6. a) Show that the exponent of convergence ρ_1 of the zeros of $\cos z$ is 1. 2

b) State Hadamard's factorization theorem.
Show that if ρ is not an integer then $\rho = \rho_1$. 1+3

c) Find the type of $\sum_{n=1}^{\infty} \frac{z^n}{(\underline{|n})^{\alpha}}, \alpha > 0$. 3

d) State Borel's second theorem. 1
