M.A./M.Sc. Examination 2018
Semester - III
Mathematics
Course: MMO-31 (A4) (New)
(Differential Equations in Ecology)

Time: Three Hours　　　　　　　　　　　　　　　　　　Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Attempt ***any ten*** questions.

1. Find the equilibria, the yield curve, and the maximum sustainable yield for a fish population that is growing according to the following Gompertz equation and that is being harvested so that the catch per unit effort is proportional to the stock size:
$$\frac{dN}{dt} = \alpha N \,\ell n\left(\frac{k}{N}\right) - qEN.$$ 1+1+2

2. Consider the logistic equation with constant rate harvesting:
$$\frac{dN}{dt} = rN\left(1-\frac{N}{k}\right) - h.$$
Treat the harvest rate h as a bifurcation parameter. Find he critical value of h for a bifurcation to occur. Which bifurcation is this? Sketch the bifurcation diagram. What are the effects of over harvesting of the system. 1+1+1+1

3. What is Functional response? Derive the expression of Holling type-II functional response. Find the half saturation constant. 1+2½+½

4. Use centre manifold theorem to determine the qualitative behaviour of the origin for the system
$$\dot{x} = xy,$$
$$\dot{y} = -y - x^2.$$ 4

5. A biologist begins a laboratory study of a certain animal population at a time $t = 0,$ when the population consists of 18 individuals. On the basis of the data collected during the experiments, the biologist finds that the differential equation
$$\frac{dP}{dt} = \frac{1}{1000}(30-P)P$$
describes the growth and regulation of the population. Solve the above equation for the population size P as a function of t. Find the point of inflection if there be any. Sketch the graph of the function. Is there an instant of time when the growth rate of the population is maximum? 1+1+1+1

6. Find the *Z*-controller $u_{pred}(t)$ of the disease model

$$\frac{ds}{dt} = rs - bs^2 - csI - \frac{\beta sI}{a+s} - \frac{\alpha_1 sP}{e+s},$$
$$\frac{dI}{dt} = \frac{\beta sI}{a+s} - \frac{\alpha_2 IP}{d+I} - MI,$$
$$\frac{dP}{dt} = \frac{c_1\alpha_1 sP}{e+s} + \frac{c_2\alpha_2 IP}{d+I} - u_{pred}(t)P,$$

so that the infected prey population $I(t)$ achieve a desired state $I_d(t)$. 4

7. Find the necessary and sufficient condition for Hopf-bifurcation of the cooperation model by treating α as a bifurcation parameter:

$$\frac{dx}{dt} = \frac{r_0 x}{1+\rho\alpha y} - dx - ax^2 - (P+\alpha y)xy,$$
$$\frac{dy}{dt} = c(P+\alpha y)xy - my.$$

Also find the critical value of α, at which Hopf-bifurcation occurs. 3+1

8. Write the properties of fear function $f(k,v) = \frac{1}{1+kv}$, where *k* is the strength of fear effect and *v* is the predator population.
Prove that the unique positive equilibrium of the model

$$\frac{du}{dt} = \frac{r_0}{1+kv}u - du - au^2 - Puv,$$
$$\frac{dv}{dt} = cPuv - mv,$$

is globally asymptotically stable if $r_0 > d + {am}/{cP}$. 1½+2½

9. What are the advantages of a dimension-less model.
By using suitable dimensionless variables, non-dimensionalize the Hastings-Powell model

$$\frac{dX}{dT} = R_0 X\left(1-\frac{X}{k}\right) - \frac{c_1 A_1 XY}{B_1+X},$$
$$\frac{dY}{dT} = \frac{A_1 XY}{B_1+X} - \frac{A_2 YZ}{B_2+Y} - D_1 Y,$$
$$\frac{dZ}{dT} = \frac{c_2 A_2 YZ}{B_2+Y} - D_2 Z.$$

4

10. Prove that the system

$$\frac{dx}{dt} = \frac{r_0}{1+ky}x - dx - ax^2 - xy(P+\alpha y),$$
$$\frac{dy}{dt} = c(P+\alpha y)xy - my,$$

is dissipative. 4

11. Prove that the disease model with Allee phenomenon

$$\frac{ds}{dt} = rs(s-\theta)(1-s) - csI,$$
$$\frac{dI}{dt} = csI - \mu I,$$

shows bistability behaviour between equilibrium points. 4

12. Find the characteristic equation of the Jacobian matrix for the delay model

$$\frac{dx}{dt} = x(1-x) - \frac{a_1 xy}{1+b_1 x},$$
$$\frac{dy}{dt} = \frac{a_1 xy}{1+b_1 x} - \frac{a_2 yz}{1+b_2 y} - d_1 y,$$
$$\frac{dz}{dt} = \frac{a_2 y(t-\tau)z(t-\tau)}{1+b_2 y(t-\tau)} - d_2 y,$$

where τ is the delay parameter. 4

13. Prove that two systems

$$\frac{dx}{dt} = x(1-x) - \frac{a_1 xy}{1+b_1 x},$$
$$\frac{dy}{dt} = \frac{a_1 xy}{1+b_1 x} - \frac{a_2 yz}{1+b_2 y} - d_1 y,$$
$$\frac{dz}{dt} = \frac{a_2 yz}{1+b_2 y} - d_2 y,$$

and 4

$$\frac{dx}{dt} = x(1-x) - \frac{a_1 xy}{1+b_1 x},$$
$$\frac{dy}{dt} = \frac{a_1 xy}{1+b_1 x} - \frac{a_2 yz}{1+b_2 y} - d_1 y,$$
$$\frac{dz}{dt} = cz^2 - \frac{wz^2}{y+D},$$

can be synchronized via bi-directional migration.

14. Write a MATLAB code to draw the phase portrait and time-series solutions in a single figure for the system

$$\frac{dx}{dt} = ax\left(1-\frac{x}{k_1}\right) - \frac{pxy}{mx+ny},$$
$$\frac{dy}{dt} = by\left(1-{y}/{k_2}\right) + \frac{\rho pxy}{mx+ny}.$$

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