Use separate answer script for each unit

M.A./M.Sc. Examination 2018

Semester - III

Mathematics

Course: MMC-35 (New)

(Galois Theory-I and Multivariable Analysis)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I

Galois Theory-I (Marks: 2×10=20)

Answer ***any two*** questions.

1. a) Find the degree of the extension $\mathbb{Q}\left(\sqrt[3]{2},\sqrt{3}\right)$ over $\mathbb{Q}$. 2

   b) Show that a finite extension of prime degree is a simple extension. 2

   c) Let K/F be an extension and let $\alpha,\beta\in K$ are algebraic over F with degrees m,n respectively. If $(m,n)=1$ then show that $\left[F(\alpha,\beta):F\right]=mn.$ 2

   d) Show that an extension F/K is algebraic if and only if, every ring R with $K\subset R\subset F$ is a field. 4

2. a) Show that the splitting field of a polynomial of degree n over K has degree utmost $n!$. Give an example of it. 3+1

   b) Let $f(x),g(x)\in K[x]$, a polynomial ring over *K*. Suppose that $g(x)=f(ax+b)$, where $0\neq a,\ b\in K$. Prove that $f(x)$ and $g(x)$ have same splitting field over *K*. 2

   c) Examine if $x^4+x+1\in\mathbb{Q}[x]$ is a separable polynomial. 2

   d) Let *F* be a field of characteristic *p* (*p* being prime) and K/F be a finite irreparable extension. Show that $p\,|\,[K:F]$. 2

3. a) Construct a polynomial $f(x)$ over $\mathbb{Z}_p$ (*p* is prime) such that $g(x)\,|\,f(x)$, for every irreducible polynomial $g(x)\in\mathbb{Z}_p[x]$. 4

   b) Let *F* be a field such that the multiplicative group $\left(F^*,\cdot\right)$ is cyclic. Show that *F* is finite. Find a generator of the cyclic group of nonzero elements of a finite field of order 29. 3+1

   c) Does there exist a finite field of order 70? Justify your answer. 2

Unit-II

Multivariable Analysis (Marks: 2×10=20)

Answer ***any two*** questions.

1. Suppose $f: A \to \mathbb{R}^m$ be a function and $A \subset \mathbb{R}^n$ be an open set.
   a) Define total derivative of f at $x_0 \in A$. 1
   b) If f is differentiable at $x_0 \in A$ then show that f is continuous at x_0. 3
   c) Suppose $f = (f_1, f_2, \ldots f_m)$. If all partial derivatives $\frac{\partial f_i}{\partial x_j}$ where $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$ exist and continuous in a neighbourhood of a point $x_0 \in A$, then show that f is differentiable at x_0.
   d) Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = \left(\frac{\sin x + e^y}{x^2 + y^2}, \frac{1}{x^2 + y^2 - 1} \right)$. Show that f is differentiable at every point $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 \neq 0$ or 1. 2

2. a) Suppose $f: A \to \mathbb{R}^m$ and $g: B \to \mathbb{R}^p$, where $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are open sets. Suppose further that $f(A) \subset B$. If f is differentiable at $x_0 \in A$ and g is differentiable at $f(x_0) \in B$ then show that $g \circ f$ is differentiable at x_0 and $D(g \circ f)(x_0) = (Dg)(f(x_0)) Df(x_0)$. 4
   b) State and prove implicit function theorem for the function $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. 6

3. a) State and prove Taylor's theorem for a function $f: \mathbb{R}^n \to \mathbb{R}$. 4
   b) Let $f: A \to \mathbb{R}$, $A \subset \mathbb{R}^n$ be an open set. If $x_0 \in A$ be a local minima of f then show that the Hessian matrix is $+ve$ semi definite. 3
   c) Find the volume of largest rectangular box with edges parallel to the co-ordinate axes and inscribed in the ellipsoid
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

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