B.A./B.Sc. (Honours) Examination 2018

Semester – V

Mathematics

Course: BMC-51

(Analysis-V)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings.

Answer ***any two*** questions from Group-A and ***any two*** from Group-B.

Group-A

1. a) Let $\{f_n\}$ be a sequence of differentiable functions converges pointwise to f on $[a,b]$ and $\{f_n'\}$ converges uniformly to g on $[a,b]$. Show that f is differentiable and $f' = g$ on $[a,b]$. 5

   b) Let $\{r_i\}$ be an enumeration of $Q \cap [0,1]$.
   For each $n = 1, 2, \ldots$ define

   $f_n(x) = 1$ if $x = r_1, r_2, \ldots, r_n$

   $= 0$ otherwise in $[0,1]$.

   To which function does the sequence $\{f_n\}$ converge pointwise? Is this convergence uniform? Support your answer. 3

   c) Test for convergence of the series $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$ on $\mathbb{R}$. 2

2. a) State and prove Weierstrass Approximation Theorem for a real valued continuous functions over $[a,b]$. 6

   b) State Abel's Lemma. State and prove Abel test for uniform convergence of a series of functions. 4

3. a) Define radius of convergence of a power series? State and prove Cauchy-Hadamard theorem. 4

   b) If a power series converges for $x = x_0 \neq 0$, then show that it is uniformly and absolutely convergent on any compact interval $[a,b] \subset (-|x_0|, |x_0|)$. 3

   c) If two power series $\sum a_n x^n$ and $\sum b_n x^n$ have non zero radius of convergence and if they are equal on some neighbourhood of zero, then prove that $a_n = b_n \forall n = 0, 1, \ldots$. 3

P.T.O.

Group-B

4. a) State Yong's theorem.
A function is defined as follows:

$$f(x,y)=\frac{x^2y^2}{x^2+y^2} \quad (x,y)\neq(0,0)$$
$$= \quad 0 \qquad (x,y)=(0,0)$$

Show that
i) f does not satisfy all the conditions of Young's theorem, but
ii) $f_{xy}(0,0)=f_{yx}(0,0)$. 1+2

b) Let a function f be defined as follows:

$$f(x,y)=\frac{xy}{\sqrt{x^2+y^2}}, \ (x,y)\neq(0,0)$$
$$= \quad 0 \qquad (x,y)=(0,0)$$

Show that f is continuous at $(0,0)$ and both f_x and f_y exist at $(0,0)$. Examine the differentiability of f at (0,0). 2+1

c) State and prove Mean value theorem for a function of two variables. 1+3

5. a) State Implicit function theorem for a function of two variables. 2

b) If roots of the equation $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ are $\lambda=u,v,w,$ then prove that $\frac{\partial(u,v,w)}{\partial(x,y,z)}=-\frac{2(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$. 4

c) State and prove Taylor's theorem for functions of two variables. 1+3

6. a) Find the Fourier series generated by $f(x)=|x|$ of period 2π on $[-\pi,\pi]$.
Show that the series converges to $|x|$ on $[-\pi,\pi]$ and hence prove that

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}.$$ 3+1

b) Let $f(x,y)=x^4+y^4-zx^2$. Show that f has a local minimum value at $(-1,0)$ and $(1,0)$. Also show that $(0,0)$ is a saddle point of f. 3

c) Define total derivative Df of a function $f:\mathbb{R}^2\to\mathbb{R}$ at a point $(x_0,y_0)\in\mathbb{R}$. Show that it is a linear functional on $\mathbb{R}^2$. 3