M.Sc. Examination, 2018
Semester – III
Physics
Elective Course: MPE-363
(Topics in Modern Quantum Mechanics)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Symbols have their usual meanings unless specified otherwise.
Answer ***any four*** of the following questions.

1. (a) Verify the truth in the statement involving the operators θ and $p_\theta = -i\hbar\frac{\partial}{\partial\theta}$, where θ corresponds to angular co-ordinate:
The relation $\triangle\theta \triangle p_\theta \geq \frac{\hbar}{2}$ is not valid, because the domain of p_θ and θ are not the same. 2

   (b) Consider a pure ensemble of spin-$\frac{1}{2}$ particles with $\langle S_x\rangle = \frac{1}{12}$, $\langle S_z\rangle = \frac{1}{12}$, where $\langle S_a\rangle$ denotes the ensemble average of S_a. The sign of $\langle S_y\rangle$ is known to be positive, but, its value is unknown. Construct the density matrix. 3

   (c) The first-order Born amplitude for a spherically symmetric potential can be expressed as,
$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rV(r)\sin qr\, dr, \quad q = |\,\vec{k} - \vec{k}'\,|$$
   i. Find an expression for the differential cross-section for Yukawa potential $V(r) = \frac{V_0 e^{-\mu r}}{\mu r}, \mu > 0$. 3
   ii. Use the above result to find differential cross-section for the Coulomb potential in terms of the kinetic energy of the incident particle and the angle θ between $\vec{k}$ and $\vec{k}'$. 1
   iii. Does the differential cross-section depend on θ for low energy scattering? 1

2. (a) Verify the truth in the statement: A hermitian(symmetric) operator is not necessarily self-adjoint, while a self-adjoint operator is necessarily hermitian. 2

   (b) State von Neumann's criterion for an operator to be self-adjoint and its self-adjoint extensions. 2

   (c) Determine the domain of the momentum operator in one dimension within the range $-a \leq x \leq a$ so that it is self-adjoint. 4

   (d) Consider the classical function $F = x^2 p_x$, where x and p_x are co-ordinate and momentum, respectively. Suggest a possible form for the quantum mechanical hermitian operator corresponding to F with justification. 2

3. (a) Comment on the possibility or impossibility of the existence of eigenstate of harmonic oscillator creation operator. Justify your comment with appropriate derivations. 2

   (b) Prove the completeness relation for the harmonic oscillator coherent states $|\alpha\rangle$ assuming the same relation for the number operator states: 6
$$\frac{1}{\pi}\int d^2\alpha\ |\alpha\rangle\langle\alpha| = 1$$

   (c) Show that $\langle\alpha|x(t)|\alpha\rangle$ resembles the behaviour of a classical oscillator, where x and t denote the space co-ordinate and time, respectively. 2

4. Consider the zero-energy ground-state wave-function ψ_0 of a Hamiltonian H,

$$\psi_0(x) = exp(-Ax + \frac{B}{\alpha}e^{-\alpha x}),\ (A, B, \alpha) \in \Re.$$

(a) Find condition(s) for normalizability of ψ_0 on the whole line. 2

(b) Find an expression of the Hamiltonian H. 2

(c) Find a Hamiltonian that is isospectral with H except for the ground-state. 1

(d) Solve the eigenvalue problem of H by using the techniques of supersymmetric quantum mechanics. 5

5. (a) Give arguments in favour of the statement: The matrix representation of the spin-operator S_a for a spin-1 particle satisfies the relation $S_a^3 = S_a, a = x, y, z$. 2

(b) Consider the spin-singlet state $|0,0\rangle$ of two spin-$\frac{1}{2}$ operators $\vec{S}_{(A)}$ and $\vec{S}_{(B)}$, the 2×2 identity matrix I and two linearly independent unit vectors $\hat{a}, \hat{b}$. Prove the following identity: 4

$$\left\langle 0,0 \left| \left(\frac{1}{2}I + \vec{S}_{(A)} \cdot \hat{a}\right)\left(\frac{1}{2}I + \vec{S}_{(B)} \cdot \hat{b}\right) \right| 0,0 \right\rangle = \frac{1}{4}\left(1 - \hat{a} \cdot \hat{b}\right)$$

(c) Determine whether the state $|\psi\rangle$ involving two spin-$\frac{1}{2}$ particles is entangled or not by calculating its Schmidt number: 4

$$|\psi\rangle = \frac{1}{\sqrt{2}}[\ |S_z;+\rangle_A\ |S_z;+\rangle_B + |S_z;-\rangle_A\ |S_z;-\rangle_B]$$

6. Consider the Hamiltonian $H = \frac{B}{2}\hat{n} \cdot \vec{\sigma}$, where $\hat{n}$ is the unit vector on the Bloch sphere and components of $\vec{\sigma}$ are the three Pauli matrices.

(a) Find the eigenstates and the corresponding eigenvalues of H. 4

(b) Derive an expression for the Berry's phase for the ground-state. 6

7. (a) Evaluate the propagator $K(x_f, t_f; x_i, t_i)$ of a free particle of mass m in the momentum basis and show that, 3

$$K(x_f, t_f; x_i, t_i) = \sqrt{\frac{m}{2\pi i\hbar(t_f - t_i)}}e^{\frac{i}{\hbar}S_0[x_c]},\ S_0[x_c(t)] \equiv \frac{m}{2}\frac{(x_f - x_i)^2}{t_f - t_i}.$$

(b) Verify that the propagator $K(x_f, t_f; x_i, t_i)$ satisfies the free particle Schrödinger equation in terms of the variables x_f and t_f. 2

(c) Evaluate $K(x_f, t_f; x_i, t_i) = \mathcal{N} \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}[x(t)]e^{\frac{i}{\hbar}S_0[x(t)]}$ by using path-integral technique and compare it with the result obtained for the same in the momentum basis to fix the constant $\mathcal{N}$. 5