B.Sc. (Honours) Examination 2018

Semester – III (CBCS)
Mathematics
Course: CC-5
(Analysis-III)

Time: Three Hours Full Marks: 60

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any six*** questions.

1. a) Let $A, D \subset \mathbb{R}$ and $f : A \to \mathbb{R}$, $g : D \to \mathbb{R}$ where $f(A) \subset D$. If c is a limit point of A and $\lim_{x \to c} f(x) = \ell$ then prove the following:
   (i) if $\ell \in D$ and g is continuous at ℓ then $\underset{x \to c}{Lt}(g \circ f)(x) = g(\ell)$
   (ii) if $\ell \notin D\left(\ell \in D'\right)$ and $\underset{y \to \ell}{Lt}\, g(y) = m$ then $\underset{x \to c}{Lt}(g \circ f)(x) = m$. 5

   b) If a function $f : [a,b] \to \mathbb{R}$ is monotone, then prove that the set of points of discontinuities of f in $[a,b]$ is countable. 5

2. a) When is a function $f : [a,b] \to \mathbb{R}$ said to be bounded variation? If f and g are two functions of bounded variation over $[a,b]$ then prove that $f + g$ and fg are of bounded variation over $[a,b]$. 6

   b) Examine if the function f defined by
$$f(x) = \begin{cases} x \sin \dfrac{\pi}{x}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$$ 4
   is of bounded variation over $[0,1]$.

3. a) Define Dedekind's section of real numbers. If (L, U) denotes the section of a real number α, then find the section of $-\alpha$ with proper justifications. 2+3

   b) Using Dedekind's section of real numbers prove that every non-empty subset of the set of real numbers which has an upper bound has the least upper bound. 5

4. a) Prove that countable union of countable sets is countable. 4

   b) Show that the set of all irrational numbers in $(0,1)$ is uncountable. 3

   c) Which of the following sets are countable? Support our answer.
   (i) $\bigcup_{n=1}^{\infty}(n, n+2)$ (ii) $(0,1) - Q$ (iii) $Q - (0,1)$ 3

5. a) Let f be continuous over $[a,b]$ and g be a non-decreasing function over $[a,b]$. Prove that $(RS)\int_a^b fdg$ exists and $(RS)\int_a^b f\,dg = (RS)\underline{\int_a^b} f\,dg + (RS)\overline{\int_a^b} f\,dg$. 4

P.T.O.

b) If g is continuous and f is non-decreasing over $[a,b]$ then show that $\exists\xi\in[a,b]$ such that $(RS)\int_a^b f\,dg = f(a)(RS)\int_a^{\xi} dg + f(b)(RS)\int_{\xi}^{b} dg$ 3

c) Evaluate $(RS)\int_0^2 x\,d(\cos x)$. 3

6. a) Test the convergence of the following (any two): 3+3

(i) $\frac{1+2}{2^3}+\frac{1+2+3}{3^3}+\frac{1+2+3+4}{4^3}+\dots$

(ii) $\sum_{n=1}^{\infty} u_n$ where $u_n=\sqrt{n^2+1}-\sqrt{n^2-1}$

(iii) $\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\dots$

b) Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms of real numbers and $\overline{\lim_{n\to\infty}} u_n^{1/n} = r$. Show that $\sum_{n=1}^{\infty} u_n$ is convergent if $r<1$ and is divergent if $r>1$. 4

7. a) Discuss the convergence of $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p},\ \ p>0.$ 3

b) If $\sum_{n=1}^{\infty} u_n$ is a convergent series, where $u_n \ge 0$ and $\{u_n\}$ is a monotone decreasing sequence then show that $\lim_{x\to\infty} nu_n = 0.$ 4

c) If $\sum_{n=1}^{\infty} u_n$ be a series of positive terms of real numbers and $v_n=\frac{u_1+u_2+\dots+u_n}{n}$, prove that $\sum_{n=1}^{\infty} v_n$ is divergent. 3

8. a) Define a function $f:[0,2]\to\mathbb{R}$ by

$$f(x)=\begin{cases} x+x^2, & \text{when } x \text{ is rational} \\ x^2+x^3, & \text{when } x \text{ is irrational}\end{cases}$$ 3

Examine the $R-$ integrability of f.

b) Let f be a real valued function bounded on $[a,b]$. Show that f is $R-$ integrable on $[a,b]$ iff $\lim_{\|P\|\to\theta}\omega(P,f)=0$. 4

c) Show that the following function *f* defined by

$$f(x)=\begin{cases}\dfrac{1}{2^n} & \text{when } \dfrac{1}{2^{n+1}}<x\le\dfrac{1}{2^n}\,(n=0,1,2,\ldots)\\ 0 & \text{when } x=0\end{cases}$$

is $R-$ integrable on $[0,1]$. Find $\int_0^1 f(x)\,dx$. 3

9. a) If *f* and *g* are two real valued functions, both are bounded and $R-$ integrable on $[a,b]$, then show that *fg* is also $R-$ integrable on $[a,b]$.
Is the converse true? Justify your answer. 3+1

b) Show by an example that the relation $\int_a^b f'(x)\,dx = f(b)-f(a)$ is not always true, for a real valued function *f* over $[a,b]$. 3

c) Show that $\dfrac{1}{a}\le\int_0^1\dfrac{\sin\pi x}{1+x^2}\,dx\le\dfrac{2}{\pi}$. 3