B.Sc. (Honours) Examination 2018

Semester – III (CBCS)

Mathematics

Course: CC-6

(Algebra-II)

Time: Three Hours | Full Marks: 60

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***all three*** questions.

1. Answer ***any two*** questions: 2×10=20

a) i) In the group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, find all cosets of the subgroup $H = \{z \in C^* : |z| = 1\}$ and interpret geometrically. 3+2

ii) Let G be a finite group of order n and $a \in G$. Then show O(a) divides n and $a^n = e$. 2

iii) Let G be a group of order less than 320. Suppose G has subgroups of order 35 and 45. Find the exact order of G. 3

b) i) Let H be a subgroup of a group G. If $x^2 \in H$ for all $x \in G$, then show that H is a normal subgroup of G and G/H is commutative. 1+2

ii) Let H be a normal subgroup of a group G such that H contains only two elements. Show that $a, b \in G$. 3

iii) Give an example of a group for which a subgroup of it is not normal. Justify your answer. 2

iv) Let $G = \langle a \rangle$ be the cyclic group such that $O(a) = 12$. Let $H = \langle a^4 \rangle$. Find the order of $a^3 H$ in G/H. 2

c) i) State and proof Cauchy's theorem for finite groups. 1+4

ii) Let G be a group of order H where p is prime. Write the class equation for G. 3

iii) Let G be a commutative group of order 15. How many subgroups of order 5 are there in G? 2

d) i) Let $\Pi = (1\,2\,3) \in S_4$. Find a conjugate of Π. Give an example of an element of order 2 which is not a 2-cycle in S_4. Justify your answer. 2+2

ii) Show that the number of even permutations in $S_n, (n \geq 2)$ is the same as the number of odd permutations. 4

iii) Find the number of distinct 5-cycles in S_7. 2

P.T.O.

2. Answer ***any two*** questions: 2×10=20

a) i) Let R be a finite ring without zero divisors. Then show that R contains unity. Give an example to show that the result does not hold for an infinite ring without having zero divisors. 4+1

ii) Prove that every maximal ideal in a commutative ring with unity is prime. If the condition of commutativity is dropped, does the result hold? Justify your answer. 4+1

b) i) Show that the field $\mathbb{Q}$ of rational numbers has no proper subfields. 4

ii) Give an example of a ring R and a subring S of R having different unities. 4

iii) Let R be a ring with unity, and $a \in R$ be a nilpotent in R. Show that $1+a$ is a unit in R. 2

c) i) Show that characteristic of a finite integral domain is prime. Give an example of an integral domain which have finite characteristic. Justify your answer. 3+2

ii) Show that the set of all nilpotent elements in a commutative ring R forms an ideal of R. 2

iii) Let I be an ideal of R. Show that $(I,+)$ is a normal subgroup of $(R,+)$. 3

d) i) Show that the ideal $M_c = \{f(x) \in C[0,1] : f(c) = 0\}$ is a maximal ideal in the ring $C[0,1]$. 3

ii) Let R be an integral domain. Show that if every ideal of R is prime, then R is a field. Hence show that $\mathbb{Z}$ is not a field. 2+1

iii) Let I be an ideal of a ring R with unity. Prove that R/I is a skewfield if and only if I is maximal. 4

3. Answer ***any two*** questions: 2×10=20

a) i) Is the set of all positive real numbers $\mathbb{R}^+$ a vector space over $\mathbb{R}$ with respect to usual addition in $\mathbb{R}^+$ and scalar multiplication defined by $cx = x^c$ for $x \in \mathbb{R}^+$, $c \in \mathbb{R}$? Justify your answer. 3

ii) Let V be a vector space over F. Show that V is also a vector space over any subfield of F. 3

iii) Let W_1 and W_2 be two subspaces of a vector space V. Show that $W_1 + W_2$ is the subspace of V generated by $W_1 \cup W_2$. 4

b) i) Let S be a linearly independent subset of a vector space V, and $v \in V \setminus S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in span(s)$. 4

ii) Let $S = \{(1,2),(2,1),(1,0),(5,3)\} \subset \mathbb{R}^2$. Find a smallest subset T of S such that $L(S) = L(T)$. 3

iii) Let W be a subspace of V. Show that there exist a subspace W' of V such that $V = W \oplus W'$. 3

c) i) Extend the set $S = \{(1,-2,5,-3),(0,7,-9,2)\}$ to a basis of $\mathbb{R}^4$. 4

ii) Give two different bases of $\mathbb{Z}_5^3$ over $\mathbb{Z}_5$. 3

iii) Show that every finitely generated vector space has a basis. 3

d) i) Let V be an inner product space over a field F. For $x, y \in V$, show that $|\langle x, y\rangle| \le \|x\|.\|y\|$. 2

ii) Consider the subspace U of $\mathbb{R}^4$ spanned by the vectors:

$$v_1 = (1,1,1,1),\ v_2 = (1,1,2,4),\ v_3 = (1,2,-4,-3).$$

Find an orthonormal basis of U. 4

iii) Let β be a basis for a subspace W of an inner product space V, and let $z \in V$. Prove that $z \in W^{\perp}$ if and only if $\langle z, v\rangle = 0$ for every $v \in \beta$. 4

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