

Use Separate Answer
Scripts for each Unit

Undergraduate Examination 2018
Semester – I (CBCS)
Mathematics
Generic Elective Course: GEC-1
(Differential Calculus, Integral Calculus & Vector Calculus)

Time: Three Hours

Full Marks: 60

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

UNIT-I (FM:20)
(Differential Calculus)
Answer **any two** questions.

1. a) State Rolle's theorem and interpret it geometrically. 1+2=3
b) State Lagrange's Mean value theorem and use it to show that
$$2x < \log\left(\frac{1+x}{1-x}\right) < \frac{2x}{1-x^2}; \quad 0 < x < 1. \quad \text{1+4=5}$$

c) Evaluate $\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x}$. 2
2. a) If 40ft of sheet metal are to be used in the construction of an open tank with square base then calculate the dimensions so that the capacity is maximum. 4
b) Apply Mclaurin's theorem to expand $\log(1+x)$. 3
c) Find the asymptotes of the cubic
$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0 \quad \text{3}$$
3. a) When is a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ said to be homogeneous of degree n ? 1+2=3
Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = x^n \log\left(\frac{y}{x}\right)$, $n \in \mathbb{N}$.
Use Euler's theorem to compute the value of $\left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right)$.
b) Write the conditions that a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ has to satisfy for
 $f_{xy}(a, b) = f_{yx}(a, b)$; for some point $(a, b) \in \mathbb{R}^2$.
Consider
 $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by,
$$f(x, y) = \begin{cases} x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right); & xy \neq 0 \\ 0 & \text{when } xy = 0 \end{cases}$$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. 1+4=5
c) Prove that the curvature of a circle is constant. 2

UNIT-II (FM:20)

(Integral Calculus)

Answer **any two** questions.

1. a) Prove that $\sqrt{\pi} \sqrt{2x} = 2^{2x-1} \sqrt{x} \left(x + \frac{1}{2}\right)$, for $x > 0$.

and show that

$$\int_0^t x^{k-\frac{1}{2}} (t-x)^{k-\frac{1}{2}} dx = \frac{\pi}{2^{4k-1}} \frac{(2k-1)!}{k!(k-1)!} t^{2k}, k \in \mathbb{N}. \quad 6+2$$

- b) Using the formula $\overline{m} \overline{1-m} = \frac{\pi}{\sin m\pi}$ $0 < m < 1$, prove that $\overline{\frac{1}{2}} = \sqrt{\pi}$ and then show

$$\text{that } \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \quad 2$$

2. a) Find the reduction formula for $\int \sin^m x \cos^n x dx$, where m, n are positive integers greater than 1. Also show that

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1 & \text{if } n \text{ is odd} \end{cases}. \quad 4+3$$

- b) Find the entire area enclosed by the circle $r = 2a \sin \theta$. 3

3. a) Find the volume and surface area of the solid generated by revolving one arc of the cycloid

$$x = a(\theta + \sin \theta) \quad 2+3$$

$$y = a(1 + \cos \theta)$$

about its base.

- b) Find the perimeter of the cardioid $r = a(1 - \cos \theta)$ and show that the intrinsic equation is of the form $s = 4a\left(1 - \cos \frac{\psi}{3}\right)$. 3+2

UNIT-III (FM:20)

(Vector Calculus)

Answer **any two** questions.

7. a) If \vec{f} is differentiable at t_0 , then show that it is continuous at t_0 .

Is the converse true? Justify your answer. 4

- b) If $\vec{r} = a \cos t \hat{i} + a \sin t \hat{j} + bt \hat{k}$, find the value of $\left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)^2$ 3

- c) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ and $f(r)$ is a scalar function possessing first and second order derivatives, prove that $\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r} \frac{df}{dr}$. 3

8. a) Find the maximum value of the directional derivative of the scalar point function

$$\phi(x, y, z) = x^2 - y^2 + z^2 \text{ at } (1, 3, 2).$$

Also find the direction in which it occurs. 4

- b) If the vectors $\vec{\alpha}$ and $\vec{\beta}$ are irrotational, then show that $\vec{\alpha} \times \vec{\beta}$ is solenoidal. 3

- c) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ is orthogonal to the surface $4x^2y + z^3 = 4$ at $(1, -1, 2)$. 3

9. a) If $\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C \vec{A} \cdot d\vec{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along

the path $C : x = t, y = t^2, z = t^3$. 3

- b) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is conservative force field.

Find also the work done by the force on a particle moving in the field from the point $(1, -2, 1)$ to the point $(3, 1, 4)$. 4

- c) Evaluate $\oint_C \{(\cos x \sin y - xy) dx + \sin x \cos y dy\}$ by Green's theorem. 3