B.Sc. (Honours) Examination 2018

Semester – I (CBCS)
Mathematics
Core Course: CC-1
(Analysis-I)

Time: Three Hours **Full Marks: 60**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any six*** questions.

1. a) State completeness property of $\mathbb{R}$ and hence deduce that every nonempty subset of $\mathbb{R}$ which is bounded below has an intimum. 4

   b) Find the 'sup' and 'Inf' of the set $\{x \in \mathbb{R} : 3x^2 + 8x - 3 < 0\}$. 2

   c) State Archimedean property of $\mathbb{R}$. Prove that if $x \in \mathbb{R}$ and $x > 0$, then $\exists$ a natural number m such that $m - 1 \le x < m$. 4

2. a) State and prove Bolzano-Weierstrass theorem for a set. 1+4

   b) Show that for any two subsets A, B of $\mathbb{R}$, $(A \cap B)' \subset A' \cap B'$. Does the equality hold? Give reasons. 5

3. a) If $G \subset \mathbb{R}$ is an open set, then prove that $\mathbb{R} \setminus G$ is closed set. 2

   b) State and prove Darboux's theorem. 5

   c) Let $f : [-1,1] \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} 0 & if \ x \in [-1,0] \\ 1 & if \ x \in (0,1] \end{cases}$$

   Does there exist a function g such that $g'(x) = f(x) \forall x \in [-1,1]$? 3

4. a) State and prove Lagrange's MVT. Give its geometrical interpretation. 5

   b) Show that $\dfrac{2x}{\pi} < \sin x < x$ for $0 < x < \pi/2$. 3

   c) Show that between any two real roots of $e^x \cos x + 1 = 0$ there is at least one real root of the equation $e^x \sin x + 1 = 0$. 2

5. a) If $y = \cos(m \sin^{-1} x)$, prove that

   $(1 - x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0$. Hence find $(y_n)_0$. 5

   b) Find the asymptotes of the curve

   $x^3 + x^2 y - xy^2 - y^3 + x^2 - y^2 = 2$. 3

   c) Find the points of inflexion, if any, of the curve $y = x^3$. 2

P.T.O.

6. a) If $\underset{x\to a}{Lt}\, f(x)=\ell$ and $\underset{x\to a}{Lt}\, g(x)=m$ then prove that $\underset{x\to a}{Lt}\, f(x)g(x)=\ell m$. 4

   b) Using $(\varepsilon-\delta)$ definition evaluate $\underset{x\to 0}{Lt}\, x\sin\frac{1}{x}$. 2

   c) Let $f:[a,b]\to[c,d]$ and $g:[c,d]\to[e,f]$ be two continuous mappings. Prove that $g\circ f:[a,b]\to[e,f]$ is continuous. 4

7. a) If a sequence $\{x_n\}$ converges to ℓ then show that the sequence $\{y_n\}$ where $y_n=\frac{x_1+x_2+\ldots+x_n}{n}$ converges to ℓ. 5

   b) Prove that every Cauchy sequence of real numbers is bounded. 3

   c) Examine if the sequence $\{x_n\}$ where $x_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ is a Cauchy sequence. 2

8. a) Let $\{[a_n,b_n]\}$ be a sequence of closed bounded intervals of $\mathbb{R}$ such that $[a_{n+1},\, b_{n+1}]\subset[a_n,\, b_n]$ for $n=1,2,3,\ldots$ and $b_n-a_n\to 0$ as $n\to\infty$. Prove that $\exists$ unique ℓ such that $\ell\in[a_n,b_n]\,\forall n=1,2,3,\ldots.$. 6

   b) Prove that the sequence $\{x_n\}$ defined by $x_1=\sqrt{7}$ and $x_{n+1}=\sqrt{7+x_n}$ for all $n\geq 1$ converges to the positive root of the equation $x^2-x-7=0$. 4

9. a) Obtain a reduction formula for $\int\cos^m x\sin nx\,dx$ and deduce the value of $\int_0^{\pi/2}\cos^5 x\sin 3x\,dx.$. 6

   b) State L. Hospital's theorem and evaluate $\underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^{1/x}$. 4
