M.Sc. Examination, 2018
Semester-I
Physics (Core)
Course: MPC-14
(Quantum Mechanics-I)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin
Answer ***any four*** *questions*
Symbols bear their usual meanings

1. a) Show that if $\hat{A}$ is Hermitian and $\lambda \in \mathbb{R}$, the operator $e^{i\lambda\hat{A}}$ is unitary.
b) State the properties that must be satisfied for a given space to be a Hilbert space.
c) The operator $\hat{T}$ is defined as $\hat{T} = \alpha\hat{Q}^{\dagger}\hat{Q}$ where α = a real number and $\hat{Q}$ is an operator (not necessarily Hermitian). Show that $\hat{T}$ is Hermitian.
d) Show that if two operators are Hermitian and their product is also Hermitian, then the operators commute.
e) Show that the eigen values of an unitary operator are complex numbers of moduli equal to one. 2+2+2+2+2=10

2. a) The basis vectors of a representation are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. Construct a transformation matrix U for transformation to another representation having basis vectors $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$ and $\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$.
b) Find the matrix representation of $\langle\Psi|\,\hat{A}|\,\varphi\rangle$.
c) (i) Use the separation of variable technique to obtain solution for a 3-dimensional isotropic harmonic oscillator.
(ii) Find the total energy and degeneracy of the second excited state. 3+2+(3+2)=10

3. The Hamiltonian operator for a linear harmonic oscillator is given by
$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{X}^2.$$
Define $\hat{q} = \sqrt{\frac{m\omega}{\hbar}}\hat{X},\ \hat{p} = \frac{\hat{P}}{\sqrt{m\omega\hbar}}$ *and* $\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$.

a) Prove that $[\hat{a}, \hat{a}^{\dagger}] = 1$.
b) Express $\hat{H}$ in terms of $\hat{a}$ and $\hat{a}^{\dagger}$.
c) Let the normalized eigenstates of $\hat{H}$ be denoted by $|\,n\rangle$. Show that $\hat{a}|\,n\rangle = \sqrt{n}|\,n-1\rangle$
d) From these, show that the ground state wave function for harmonic oscillator in position space is given by $\Psi_0(x) = \langle x|0\rangle = Ae^{-\frac{x^2}{2x_0^2}}$ where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$.
e) Also find the normalization constant A appearing in (d). 1+2+3+2+2=10

4. a) State and prove Schwartz inequality.
b) For the angular momentum operator given by
$\hat{J}^2|\,j\ m\rangle = \hbar^2 j(j+1)|\,j\ m\rangle$
$\hat{J}_z|\,j\ m\rangle = m\hbar\,|\,j\ m\rangle$
establish the following relations:
(i) $[\hat{J}_z, \hat{J}_{\pm}] = \pm\hbar\hat{J}_{\pm}$
(ii) $\hat{J}^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_z^{\,2}$
c) Show that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ where σ's are Pauli spin matrices. 3+(2+3)+2=10

P.T.O.

5. a) Consider the radial equation for the hydrogen atom

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{2mr^2}{\text{h}^2}\{V(r)-E\}=l(l+1)$$

where symbols have their usual meaning. Obtain bound state solution for the above equation in asymptotic limits.

b) Express the momentum operator $\hat{p}_x$ in position representation.

c) Show that the outer product of two vectors $|\,\alpha\rangle$ and $|\,\beta\rangle$ is a linear operator.

d) What will be the orthonormality condition of the state vectors in continuous basis?

4+3+2+1=10

6. a) For any two vectors $\vec{A}$ and $\vec{B}$, which commute with $\vec{\sigma}$, show that

$$(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B})=(\vec{A}.\vec{B})+i\vec{\sigma}.(\vec{A}\times\vec{B})$$

b) A particle of mass m, which moves freely inside an infinite potential well of length $a=20$ cm, has the following wave function at $t=0$:

$$\psi(x,0)=\sqrt{\frac{3}{5}}\varphi_1(x)+\sqrt{\frac{3}{10}}\varphi_3(x)+\frac{1}{\sqrt{10}}\varphi_5(x)$$

where $\varphi_n(x)=\sqrt{\frac{2}{a}}\,sin\frac{n\pi x}{a}$.

(i) Find the wave function at any later time t.

(ii) Calculate the probability of finding the particle in state $|\,\varphi_3\rangle$ with energy E_3.

(iii) Find the average energy of the system.

3+(3+2+2)=10

7. a) Suppose a spin $\frac{1}{2}$ particle is in the state

$$|\chi\rangle=\frac{1}{\sqrt{6}}\begin{pmatrix}1+i\\2\end{pmatrix}$$

Find the expectation values of $\widehat{S_x}$, $\widehat{S_y}$ and $\widehat{S_z}$.

b) Obtain the expressions for $\widehat{L_+}$ and $\widehat{L_z}$ in spherical polar coordinates. Given

$$\hat{\imath}=\hat{r}sin\theta cos\varphi+\hat{\theta}cos\theta cos\varphi-\hat{\varphi}sin\varphi$$
$$\hat{\jmath}=\hat{r}sin\theta sin\varphi+\hat{\theta}cos\theta sin\varphi+\hat{\varphi}cos\varphi$$
$$\hat{k}=\hat{r}cos\theta-\hat{\theta}sin\theta$$

c) Consider a wave function for a system in the state

$$\psi(\theta,\varphi)=\frac{1}{\sqrt{5}}Y_{1,-1}+\sqrt{\frac{3}{5}}Y_{1,0}+\frac{1}{\sqrt{5}}Y_{1,1}=\frac{1}{\sqrt{5}}\,|\,1-1\rangle+\sqrt{\frac{3}{5}}\,|\,1\;0\rangle+\frac{1}{\sqrt{5}}\,|\,1\;1\rangle$$

Find $\langle\Psi|\,\widehat{L_+}\,|\,\Psi\rangle$.

3+(3+1)+3=10