Five Year Integrated M.Sc. Examination, 2017

Semester-II

Course: MT-1-2-1

(Mathematics-II)

Time: Three Hours **Full Marks: 60**

Questions are of value as indicated in the margin.

Answer ***any four*** questions.

1. a) Show that $\log(1+x)$ lies between $x-\frac{x^2}{2}$ and $x-\frac{x^2}{2(1+x)}$ for all $x>0$. 4

   b) If the area of a circle increases at a uniform rate, show that the rate of increase of the perimeter varies inversely as the radius. 3

   c) If *f* is continuous in $a \le x \le b$ and $f'(x) > 0$ in $a < x < b$, then show that *f* is strictly increasing in [*a, b*]. 3

   d) If $y = x^{2n}$, *n* is a positive integer, then show that $y_n = 2^n \{1\cdot 3\cdot 5 \ldots\ldots (2n-1)\} x^n$. 5

2. a) Show that $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, where $0 < \alpha < \theta < \beta < \pi/2$. 4

   b) If $f(x)$ and $g(x)$ are differentiable in [*a, b*], then show that there exists a number $\xi\,(a < \xi < b)$ such that

   $$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix}.$$ 4

   c) Show that the Cauchy's remainder after *n* terms in the expansion of $\log(1+x)$ in power of *x* is $(-1)^{n-1}\frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$, where $0<\theta<1$. 3

   d) Expand $\sin x$ in a finite series in power of *x*, with remainder in Lagrange's form. 4

3. a) If $u = \log\left(x^3+y^3+z^3-3xyz\right)$, then show that

   (i) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ (ii) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}$. 5

   b) If $z = f(u, v)$, where $u = x^2-2xy-y^2$ and $v = y$, then show that

   $(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=0$ can be transformed into $\frac{\partial z}{\partial v}=0$. 3

   c) Examine whether the function $\left(\frac{1}{x}\right)^x$ possesses a maximum or a minimum value. 4

   d) Show that the largest rectangle with given perimeter is a square. 3

4. a) Show that the portion of the tangent at any point of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ intercepted between the axes is of constant length. 4

   b) Assuming the convergence of the integral $\int_0^{\pi/2}\log\sin x$, evaluate the integral. 3

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c) Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2} dx.$ 3

d) If $I_n = \int_0^{\pi/2} \sin 2nx \cot x dx \,(n \geq 1)$, then show that $I_{n+1} = I_n$. Hence deduce $I_n = \frac{\pi}{2}$. 4+1

5. a) Prove that the vectors $2\hat{i} + 3\hat{j} - 6\hat{k}$, $6\hat{i} - 2\hat{j} + 3\hat{k}$ and $2\hat{i} - 6\hat{j} + 3\hat{k}$ form the sides of an equilateral triangle. 3

b) Show by vector method that the medians of a triangle are concurrent. 4

c) Find the value of x for which the vectors $x\hat{i} - u\hat{j} + 5\hat{k}$, $\hat{i} + 2\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} + \hat{k}$ are coplanar. 4

d) Given $\vec{\alpha} = 3\hat{i} - \hat{j}$, $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$; express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$. 4

6. a) If $f(s)$ is the Laplace transform of $F(t)$ and a is any real or complex number, then show that $f(s+a)$ is the Laplace transform of $e^{-at}F(t)$. 4

b) If $L\{F(t)\} = f(s)$, then show that

$L\{F(at)\} = \frac{1}{a} f\left(s/a\right)$, where L stands for Laplace transform. 4

c) Show that $L\{e^{at}\} = \frac{1}{s-a}$ if $s > a$, where L stands for Laplace transform. 3

d) If $f(s)$ is the Fourier transform of $F(x)$, then show that $e^{-ias} f(s)$ is the Fourier transform of $F(x-a)$. 4
