B.Sc. (Honours)Examination, 2018

Semester–V

Physics (Core)

Paper: BPC-51

(Mathematical Methods-III)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Symbols bear their usual meanings. Einstein Summation convention is used wherever applicable.

Answer***anyfour*** questions

1. A differential equation is given by
$$xy''(x) - (1-2x)y'(x) - (1-x)y(x) = 0.$$
Show that $x_0 = 0$ is an apparent singularity of the differential equation. Solve the equation by the Frobenius-Fuchs method. (2+8)

2. (a) Suppose a real function $f(x)$ can be expressed in $-1 < x < 1$ as
$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad n = 0, 1, 2, \ldots$$
Show that the coefficients c_n can be expressed as
$$c_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x)\ P_n(x)\, \mathrm{d}x.$$
(b) The recurrence relation of the coefficients of the solution of the Legendrȩ equation is given by
$$c_{r+2} = \frac{r(r+1) - n(n+1)}{(r+1)(r+2)}\ c_r, \qquad n = 0, 1, 2, \ldots$$
If $c_n = \dfrac{(2n)!}{2^n (n!)^2}$, show that
$$P_n(x) = \sum_{r=0}^{N} (-1)^r\ \frac{(2n-2r)!}{r!\ 2^n\ (n-r)!(n-2r)!}\ x^{n-2r}$$
where $N = n/2$, when $n =$ even and $N = (n-1)/2$ when $n =$ odd.
(c) Show that
$$P_n(x) = \frac{1}{2^n\ n!} \frac{\mathrm{d}}{\mathrm{d}x} \left(x^2 - 1\right)^n \qquad n = 0, 1, 2, \ldots.$$
2+4+4

3. (a) Show that
$$\frac{2n}{x}\ J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$
(b) Evaluate
$$\int x^4 J_1(x)\, \mathrm{d}x$$
(c) Show that $\cos(\frac{\pi x}{2})$, expanded upto the second power of x is given by
$$\cos\frac{\pi x}{2} = \frac{2}{\pi} + \frac{10}{\pi}\left(1 - \frac{12}{\pi^2}\right)\left(\frac{3x^2 - 1}{2}\right)$$
3+4+3

P.T.O.

4. (a) If $f(x) = \sum_n c_n\, f_n(x)$, show that the Fourier transform of $f(x)$ is given by

$$\mathscr{F}\,[f(x)] = F(k) = \sum_n c_n F_n(k)$$

where $F_i(k) = \mathscr{F}\,[f_i(x)]$.

(b) If $\mathscr{F}\,[f(x)] = F(k)$, show that

$$\mathscr{F}\,[f(x)\cos(ax)] = \frac{1}{2}\,[F(k+a) + F(k-a)]\,.$$

(c) Arrive at the expression of Fourier sine transform from Fourier transform. Find the Fourier sine transform of the function $f(x) = \mathrm{e}^{-ax}/x$. 2+3+(2+3)

5. (a) If $g(\omega) = \mathscr{F}[f(t)]$, show that $g(-\omega) = -g^*(\omega)$ is a necessary and sufficient condition for $f(t)$ to be pure imaginary.

(b) Find the Fourier sine transform of the following function

$$f(x) = \begin{cases} x^2, & \text{when } 0 \le x \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

(c) Show that the Fourier transform of the odd function

$$f(x) = \begin{cases} \mathrm{e}^{-ax} & x > 0 \\ -\mathrm{e}^{ax} & x < 0 \end{cases}$$

where $(a > 0)$, is given by

$$F(k) = \mathscr{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\frac{-2ik}{a^2 + k^2}.$$

Comment on the result. 2+3+ (4+1)

6. (a) Show that the ordinary derivative of a tensor does not retain tensorial characteristic.

(b) Show that

$$\nabla_\nu\, A^\mu = \partial_\nu\, A^\mu + \Gamma^\mu_{\alpha\nu}\, A^\alpha$$

(c) Show that the covariant differentiation of a tensor retains its tensorial characteristics.

(d) Show that

$$\Gamma_{\alpha|\mu\nu} = \frac{1}{2}\,(\partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu})$$

(To prove you may assume that $\nabla_\alpha g_{\mu\nu} = 0$). 2+3+3+2

7. (a) Show that the general equation governing the motion of a simple pendulum is an NDE

(b) How do you define a spontaneous singularity of an NDE? Show that the solution of

$$\frac{dy}{dx} = y^2$$

is sensitive to initial condition, while the solution of

$$\frac{dy}{dx} + \frac{y}{x-1} = 0$$

is not so. 3+2+5
