B.Sc. (Honours) Examination, 2019
Semester-II
Statistics
Course : CC-4
(Algebra)

Time : 3 Hours **Full Marks : 60**

Questions are of value as indicated in the margin

Group - A (Answer **any ten** questions) $10\times1=10$

1. Answer the following questions with proper justification.

   (a) What is the degree of the polynomial obtained by the sum of two different polynomials of degree m and n?

   (b) Is x^2+x+1 a factor of $x^{20}+x^7+1$?

   (c) If a polynomial of degree 4 has a root $(\sqrt{2}+i)$, then find the polynomial.

   (d) If any Sturm's function has all complex roots then what can we conclude?

   (e) Find the number of special roots of $x^{24}-1=0$.

   (f) Express $A=\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ as the product of elementary matrices.

   (g) If A, B are equivalent square matrices then show that A^2, B^2 may not be equivalent.

   (h) Give two examples of subspaces.

   (i) If W_1, W_2 are subspaces of a vector space V. Then $W_1 \cup W_2$ is a subspace of V? Justify your answer.

   (j) Write down a basis of $M_{3\times3}(\boldsymbol{R})$.

   (k) What can you say about the eigenvalues of a real orthogonal matrix?

   (l) When a matrix A of order n is diagonalisable?

   (m) Verify Dimension Theorem for the linear map $T : M^{n\times n}(F) \to F$ defined by $T(A)=tr(A)$.

   (n) If A is a non-singular matrix of order 4. Then find $rank(A^3)$.

Group – B (Answer any five questions) $5\times6=30$

2. (a) If $f(x)=x^5-3x^3+10x^2$, express $f(x+3)$ as polynomial in x.
(b) Show that the roots of the equation $\frac{1}{x+a_1}+\frac{1}{x+a_2}+\cdots+\frac{1}{x+a_n}=\frac{1}{x}$ are all real. Here a_i's are all real. 3+3

P.T.O.

3. (a) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of the equation $(\beta_1 - x)(\beta_2 - x)\cdots(\beta_n - x) + c = 0$. Then find equation whose roots are $\beta_1, \beta_2, \cdots, \beta_n$.
(b) If α, β, γ are roots of the equation $x^3 + ax + b = 0$ then show that $\sum \alpha^5 = 5ab$ and $\sum \frac{\alpha}{\beta} = -3$. 3+3

4. (a) Let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, $\beta = \{1, x, x^2\}$. A linear map $T : P_2(\boldsymbol{R}) \to M_{2\times 2}(F)$ defined by $T[f(x)] = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$. Find $[T]_\beta^\alpha$.
(b) Show that the rank of a real skew-symmetric matrix cannot be 1. 3+3

5. (a) Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \boldsymbol{R}$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, a_2) = (ca_1, a_2)$. Is V a vector space over $\boldsymbol{R}$ with these operations? Justify your answer.
(b) Prove that $W_1 = \{(a_1, a_2, \cdots, a_n) \in F^n : \sum_{i=1}^n a_i = 0\}$ is a subspace of F^n, but $W_2 = \{(a_1, a_2, \cdots, a_n) \in F^n : \sum_{i=1}^n a_i = 1\}$ is not. 3+3

6. (a) The set of all $n \times n$ upper triangular matrices is a subspace W of $M_{n\times n}(F)$. Find a basis for W. What is the dimension of W?
(b) If $S = \{(x, y, z, w) \in \boldsymbol{R}^4 : 2x + y + 3z + w = 0\}$ and $T = \{(x, y, z, w) \in \boldsymbol{R}^4 : x + 2y + z + 3w = 0\}$, then find $dim(S \cap T)$. 3+3

7. (a) If λ be an eigenvalue of a real orthogonal matrix A. Prove that $1/\lambda$ is also an eigenvalue of A.
(b) Let $A_{3\times 3}$ be a non-diagonal matrix with the property $A^{-1} = A$. Then find $Trace(A)$ and $Det(A)$. 3+3

8. (a) Show that the real quadratic form $ax^2 + 2hxy + by^2$, $(a \neq 0)$ is PD iff $a > 0$, $\begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$.
(b) $A_{2\times 2}$ be a real matrix is diagonalizable if and only if i) $[Tr(A)]^2 < 4Det(A)$ ii) $[Tr(A)]^2 > 4Det(A)$ iii) $[Tr(A)]^2 = 4Det(A)$ iv) $Tr(A) = Det(A)$.
(c) Is there a linear transformation $T : \boldsymbol{R}^3 \to \boldsymbol{R}^2$ such that $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$? 2+2+2

9. (a) If $A_{m\times m}$ is idempotent then show that $Rank(A) + Rank(I_m - A) = m$.

(b) Find the inverse of the matrix $A = \begin{pmatrix} 1 & \rho & \cdots & \rho & \rho \\ \rho & 1 & \cdots & \rho & \rho \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \rho & \rho & \cdots & 1 & \rho \\ \rho & \rho & \cdots & \rho & 1 \end{pmatrix}$ 2+4

<u>*Group – C (Answer any two questions)*</u>

10. (a) Define the basis of a vector space. Show that the basis of a vector space is not unique.

(b) Let A and B be two non-singular matrices of appropriate orders. Prove that $\begin{vmatrix} A & C \\ D & B \end{vmatrix} = |A||B - DA^{-1}C| = |B||A - CB^{-1}D|$. 5+5

11. (a) Find the determinant of the following matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_1^2 & x_2^2 & \cdots & x_{n-1}^2 & x_n^2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_{n-1}^{n-2} & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_{n-1}^{n-1} & x_n^{n-1} \end{pmatrix}.$$

(b) Prove that $(A + uv')^{-1} = A^{-1} - \dfrac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}$. 5+5

12. (a) For a matrix $A_{m\times n}$, show that $Row\ rank\ of\ A = Column\ rank\ of\ A$.

(b) State and prove Cayley-Hamilton Theorem. 5+5