Five Year Integrated M.Sc. Examination, 2019

Semester-VIII

Course: MT-4-8-2A

(Analysis-IV)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Answer ***any four*** questions.

1. a) When are two norms on a linear space said to be equivalent? Prove that any two norms defined on a finite dimensional linear space are equivalent. 1+3

   b) Verify that l_∞ (sequence space) is a Banach space w.r.t. a norm to be defined by you. 4

   c) Let $(X, \|\ \|)$ be a normed linear space and $\{x_n\}$ be a sequence in *X*. Show that if $x_n \to x$ then $\|x_n\| \to \|x\|$. 2

2. a) Prove that every linear operator defined on a normed linear space is continuous iff it is bounded. 4

   b) State Riesz Lemma and hence show that if a closed unit ball in a normed linear space *X* is compact then *X* is finite dimensional. 1+3

   c) Prove that $\mathbb{R}^2$ is a normed linear space w.r.t. $\|\ \|$ defined by $\|x\| = \max\left\{\frac{|x_1|}{a}, \frac{|x_2|}{b}\right\}$ where $a, b > 0$ and $x = (x_1, x_2)$. 2

3. a) Prove that Dual space X^* of any normed linear space *X* is a Banach space. 5

   b) Let $T : C[a, b] \to \mathbb{R}$ defined by $T(x) = \int_a^b x\,dt,\ x \in c[a, b]$. Show that *T* is a bounded linear operator and find $\|T\|$. 5

4. a) Show that every inner product space is a normed linear space. 3

   b) Prove that l_2 is a Hilbert space w.r.t. an inner product to be defined by you. 4

   c) Show that inner product in a linear space is a continuous function. 3

5. a) Define an orthonormal set in an inner product space. Prove that any orthonormal set of vectors in an inner product space is linearly independent. 1+3

   b) If N_1 and N_2 are normal operators defined on a Hilbert space *H* such that either commutes with the adjoint of the other than prove that $N_1 + N_2$ and $N_1 N_2$ are normal operators on *H*. 4

   c) Prove that every self-adjoint operator is normal but converse may not be true. 2

6. a) Show that for every bounded linear operator $T : H \to H$ where *H* is a Hilbert space, its adjoint operator T^* exists and $\|T\| = \|T^*\|$. 5

   b) Define normal operator on a Hilbert space. Prove that the set of all normal operators defined on a Hilbert space *H* is a closed subset of $B(H, H)$ where $B(H, H)$ is the set of all bounded linear operators defined on *H*. 1+4

———