Five Year Integrated M.Sc. Examination 2019

Semester-VI

Course: PH-3-6-2

(Electromagnetic Theory)

Time: Four Hours **Full Marks: 80**

Questions are of value as indicated in the margin.

Answer ***Question No.1*** and ***any four*** questions from the rest.

1. Answer any **five** from the following: $5 \times 4 = 20$
   (a) What do you mean by covariance of physical laws? Show that Newton's law of motion is not covariant under Lorentz transformation.
   (b) Define 4-velocity. What is the problem if we try to use the definition of non-relativistic velocity in relativistic mechanics?
   (c) Show that the scalar potential transforms as Lorentz scalar.
   (d) Prove that the metric tensor $g^{\mu\nu}$ transforms as a rank two contravariant tensor under a coordinate transformation of the form $x^i \to x'^i$.
   (e) Define Poynting vector. How do you physically interpret Poynting vector ?
   (f) Calculate the average power radiated by a Hydrogen atom assuming electron as a classical point particle. Hence justify that the classical model of atoms is unstable according to classical electrodynamics. You may use the power radiation formula
   $$\mathcal{P} = \frac{\mu_0}{6\pi c}|\ddot{\mathbf{p}}|^2 .$$
   (g) Calculate dipole moment of a spherical shell of radius R, which carries a surface charge $\sigma = k\cos\theta$.

2. Find out the solution of Laplace equation in spherical coordinates. Laplacian operator in spherical polar coordinates is given by
$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi^2} .$$
Use the solution to calculate the potential inside a hollow sphere of radius R, whose surface potential is given by $V_0(\theta) = k\sin^2(\theta/2)$. You may use the following orthogonality relation
$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{l,l'} ,$$
where $P_l(x)$ is the Legendre polynomial of order l. (8+7)=15

3. (a) Define covariant vectors and contravariant vectors with examples.
   (b) Show that the covariant and contravariant vectors follow different transformation rules under Lorentz transformation. However both types of vectors follow similar transformation rules in case of ordinary rotation in three dimensions.
   (c) Define event interval. How do you classify them ? 4+(4+4)+3=15

P.T.O.

4. (a) Suppose $A^{\mu\nu}$ and $B_{\mu\nu}$ are symmetric and antisymmetric tensor respectively, show that $A^{\mu\nu}B_{\mu\nu} = 0$.

(b) Define 4-potential of an electromagnetic field.

(c) Using the 4-potential define electromagnetic field tensor ($F^{\mu\nu}$). Show that $F^{\mu\nu}$ is a guage invariant quantity.

(d) We know that the Maxwell's equations with source terms can be expressed as $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$. Starting form these equations derive continuity equation for 4-current (J^μ).

(e) Prove that

$$\partial^\alpha F^{\beta\gamma} + \partial^\gamma F^{\alpha\beta} + \partial^\beta F^{\gamma\alpha} = 0$$

4+2+(1+2)+2+4=15

5. Electromagnetic field tensor is given by

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{bmatrix}$$

(a) Write the transformation rules for field tensor, $F^{\alpha\beta}$ under Lorentz transformation along $+ve$ x-direction.

(b) Suppose an inertial frame S' is moving to the right at the speed v relative to S frame. $F'^{\alpha\beta}$ and $F^{\alpha\beta}$ are the field tensors as measured by the observers standing on the S' and S frames respectively. Hence, find out the relations between the components of electric and magnetic fields in these two frames.

(c) Write the Lagrangian density of an Electromagnetic field in free space. Hence express it in terms of electric and magnetic field's components.

(d) Show that $\vec{E}.\vec{B}$ is a Lorentz scalar.

2+5+4+4=15

6. (a) Prove that

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r})\,.$$

(b) Show that the retarded time equation i.e. $t_{ret} = t - \frac{|\vec{r}-\vec{r'}(t_{ret})|}{c}$ has only one physically acceptable solution.

(c) Define volume charge density (ρ) and volume current density ($\mathbf{J}$) for a single charge q, moving along a curve $\mathbf{r}(t)$ with velocity $\mathbf{v}(t)$. Show that the the retarded 4-potential for a charge q moving with velocity $\mathbf{v}$ can be expressed as

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{c}{c-\hat{\mathbf{R}}(t').\mathbf{v}(t')}\frac{1}{R(t')}\right]_{t'=t_{ret}},$$

$$\mathbf{A}(\mathbf{x},t) = \frac{q\mu_0}{4\pi}\left[\frac{c}{c-\hat{\mathbf{R}}(t').\mathbf{v}(t')}\frac{\mathbf{v}(t')}{R(t')}\right]_{t'=t_{ret}},$$

where, $\mathbf{R}(t') = \mathbf{x} - \mathbf{r}(t')$ is the difference vector between source point and field point.

5+3+(3+4)=15

7. (a) Consider the Maxwell's equation in Lorentz guage:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\mu = \mu_0 J^\mu \,.$$

Taking Fourier transformation find out the corresponding equation for Fourier components $\tilde{A}_\mu(\mathbf{x},\omega)$ where $A_\mu(\mathbf{x},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{A}_\mu(\mathbf{x},\omega) e^{-i\omega t}$.

(b) Suppose the solution of the aforesaid equation is given by

$$A_\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{J_\mu(\mathbf{x}',t_{ret})}{|\mathbf{x}-\mathbf{x}'|}\,,$$

where t_{ret} is retarded time. Now using Taylor series expansion show that, for the radiation zone i.e. $|\mathbf{x}'| << |\mathbf{x}|$, it is possible to expand the volume current density as follows

$$J_\mu(\mathbf{x}',t_{ret}) = J_\mu(\mathbf{x}',t-r/c) + \dot{J}_\mu(\mathbf{x}',t-r/c)\frac{\mathbf{x}.\mathbf{x}'}{rc} + \ldots,$$

where $r = |\mathbf{x}|$.

(c) Using electric dipole approximation find out the relation between magnetic vector potential, $\mathbf{A}(\mathbf{x},t)$ and the time derivative of electric dipole moment, $\dot{\mathbf{p}}$.

(d) Hence calculate the magnetic field, $\mathbf{B}$ due to this magnetic vector potential and identify the part of magnetic field which will dominate in radiation zone.

3+3+4+5=15