Five Year Integrated M.Sc. Examination 2019

Semester-VI

Course: PH-3-6-1

(Advanced Quantum Mechanics-I)

Time: Four Hours **Full Marks: 80**

Questions are of equal value or as indicated in the margin.

Answer ***any four*** questions.

1. (a) What do you mean by linearly independence of a set of vectors ?

   (b) Consider three row vectors : $(1,1,0)$, $(1,0,1)$ and $(3,2,1)$. Are they linearly independent? Support your answer with details.

   (c) Suppose $|u\rangle$ and $|v\rangle$ are two vectors in Hilbert space and $\langle u|v\rangle$ is the inner product between these two vectors. Hence prove that the inner product is antilinear with respect to the first factor in the inner product.

   (d) Construct an orthogonal basis in two dimensions starting with $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{B} = \hat{\imath} - 3\hat{\jmath}$. Can you generate orthonormal basis starting with these to vectors. Use Gram-Schmidt procedure to answer this question.

   (e) Define Unitary operator. Prove that unitary operators preserve the inner product between the vectors they act on. 3+(2+4)+3+5+3=20

2. (a) The *trace* of a matrix is defined to be the sum of its diagonal matrix elements

$$Tr(\Omega) = \sum_i \Omega_{ii}.$$

   Show that

   i. $Tr(\Omega\Lambda) = Tr(\Lambda\Omega)$,

   ii. $Tr(\Omega\Lambda\theta) = Tr(\Lambda\theta\Omega) = Tr(\theta\Omega\Lambda)$,

   where Ω, Λ and θ are matrices of same dimension.

   (b) Suppose $\hat{T}$ is a linear transformation or equivalently linear operator on a n- dimensional linear vector space, $\mathcal{V}$. Find out the matrix representation of $\hat{T}$ with respect to the basis vectors of this vector space, $\{|e_i\rangle\}$.

   (c) Suppose $|\alpha\rangle$ is a generalised vector in the aforesaid vector space $\mathcal{V}$, such that $|\alpha\rangle = \sum_{j=1}^{n} a_j|e_i\rangle$. Action of T on $|\alpha\rangle$ is defined as $|\alpha'\rangle = T|\alpha\rangle$. Hence show that

$$a_i' = \sum_{j=1}^{n} T_{ij}a_j,$$

   where, $a_i' = \langle e_i|\alpha'\rangle$.

   (d) The Hamiltonian operator for a two-state system is given by

$$H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

   where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combination of $|1\rangle$ and $|2\rangle$) . (2+3)+3+4+8=20

3. (a) Suppose annihilation $(\hat{a}_-)$ and creation ($\hat{a}_+$) operator are defined as $\hat{a}_\pm = \frac{1}{\sqrt{2m}}(\hat{p} \pm im\omega\hat{x})$. Calculate the commutator : $[\hat{a}_-, \hat{a}_+]$

   (b) Use $\hat{a}_\pm$ to express the Hamiltonian of a one-dimensional (1D) harmonic oscillator of mass m and natural frequency ω. Hence, show that $\hat{a}_+$ increases the energy of $|n\rangle$ by $\hbar\omega$ amount where $|n\rangle$ is the n-th eigenstate of harmonic oscillator corresponding to the energy eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$.

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(c) Derive the position space representation of ground state wave function of 1D harmonic oscillator by using the fact that $\hat{a}_-$ annihilates the ground state of harmonic oscillator. .

(d) A one-dimensional harmonic oscillator of mass m and natural frequency ω in the quantum state

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{i}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|2\rangle$$

at time $t = 0$, where $|n\rangle$ denotes the eigenstate with eigenvalue $(n + \frac{1}{2})\hbar\omega$. Calculate the expectation value $\langle\psi(t)|\hat{x}|\psi(t)\rangle$ of the position operator, $\hat{x}$ at some arbitrary time t.

3+(2+4)+6+5=20

4. (a) Consider a Hamiltonian $H = H_0 + \lambda H'$ where ($\lambda << 1$) is a small parameter. The unperturbed Hamiltonian H_0 satisfies following eigenvalue equation : $H_0|\psi_n^0 >= E_n^0|\psi_n^0 >$, where all the eigenvalues are non degenerate.

   i. Calculate the first order corrections to the n-th energy eigenvalues and n-th eigenfunctions.

   ii. Calculate the second order corrections to the n-th energy eigenvalues.

   (b) Suppose $|\psi_a^0\rangle$ and $|\psi_b^0\rangle$ are two degenerate orthogonal eigenvectors of the Hamiltonian H^0 with same eigenvalue E^0. Let the system is perturbed by a Hamiltonian H' which does not commute with H^0. So the system is now defined by the full Hamiltonian $H = H^0 + \lambda H'$, where $\lambda << 1$. Using degenerate perturbation theory calculate the eigenvalues of H. How does these eigenvalues vary with λ. For simplicity you may assume that the $\langle\psi_i^0|H'|\psi_i^0\rangle = 0$ for $i = a, b$.

   (c) Consider a charged particle in the one-dimensional harmonic oscillator potential. Suppose we turn on a weak electric field (E), so that the potential energy is shifted by an amount $V' = -qEx$. Calculate the first and second order change in the energy for the n-th eigenstate.

(5+4)+5+6=20

5. (a) The Hamiltonian of a two-level quantum system is

$$H = \frac{1}{2}\hbar\omega \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

   Find the possible initial states in which the probability of the system being in those quantum states do not change with time. Suppose the system is found in one of such initial state of H at $t = 0$, what will happen to that system if the system is perturbed by a Hamiltonian $H' = -\frac{1}{2}\hbar\omega\sigma_3$? Will it still remain in the same state?

   (b) Prove that the variational principle can only predict the upper bound of the ground state energy of a Hamiltonian.

   (c) Let us consider the time independent Schroedinger equation in one dimension :

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi,$$

   where the eigenfunction corresponding to eigenvalue E is given by $\psi(x) = A(x)e^{i\phi(x)}$. Now choosing appropriate approximation show that

$$A(x) = \frac{C}{\sqrt{p(x)}}, \quad \text{and} \quad \phi(x) = \pm\frac{1}{\hbar}\int p(x)dx,$$

   where $p(x)$ is classical linear momentum at x. Does this approximation valid for classical turning points? Explain your answer.

   (d) Use the WKB approximation to find the allowed energies (E_n) of an infinite square well with a "self", of height V_0 extending half-way across:

$$V(x) = \begin{cases} V_0, & \text{if } 0 < x < \frac{a}{2}, \\ 0, & \text{if } \frac{a}{2} < x < a, \\ \infty, & \text{otherwise.} \end{cases}$$

   Express your answer in terms of V_0 and $E_n^0 \equiv \frac{(n\pi\hbar)^2}{2ma^2}$.

(4+1)+3+6+6=20

6. (a) Ladder operators are defined as $L_{\pm} = L_x \pm iL_y$. Suppose $|\lambda, m\rangle$ is the eigenfunction of L_z and L^2 with eigenvalue $m\hbar$ and $\lambda\hbar^2$ respectively. Hence prove the following

   i. L_+ (L_+) increases (decreases) the m value by one unit.

   ii. $L^2 = L_{\pm}L_{\mp} + L_z^2 \mp \hbar L_z$

   iii. Now choosing highest weight state show $\lambda = l(l+1)$, where l is the maximum value of m.

   iv. l is either integer or half-integer.

(b) Wave function of the electron in a hydrogen atom is given by

$$\psi(\vec{r}) = \frac{1}{\sqrt{6}}\,\phi_{200}(\vec{r}) + \sqrt{\frac{2}{3}}\,\phi_{21-1}(\vec{r}) - \frac{1}{\sqrt{6}}\,\phi_{100}(\vec{r}),$$

where $\phi_{nlm}(\vec{r})$ are the eigenstates of the Hamiltonian in the standard notation. Hence calculate the expectation value of the energy in this state. Energy of the n-th eigenstate is given by $E_n = -(13.6/n^2)\,eV$ (4+4+3+3)+6=20