# Five Year Integrated M.Sc. Examination, 2019
## Semester-VI
## Course: MT-3-6-2
## ( Advanced Modern Algebra )

**Time: Three Hours** **Full Marks: 40**

Questions are of value as indicated in the margin.
Answer ***any four*** questions.

1. a) Let $H$ be a proper subgroup of a group $G$. If for each $a \in G$, there exists $b \in G$ such that $aH = Hb$, then prove that $H$ is a normal subgroup of $G$. 2

   b) Let $G = \langle a \rangle$ be a cyclic group of order 12. Let $H = \langle a^4 \rangle$. Find the order of $a^3H$ in $G/H$. 2

   c) Find all homomorphisms from $(\mathbb{Z}, +)$ onto $(\mathbb{Z}_6, +)$. 3

   d) Show that every infinite cyclic group is isomorphic to the group $(\mathbb{Z}, +)$ of all integers. 3

2. a) Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $\gcd(m, n) = 1$. 4

   b) Let $G$ be a finite group and $a \in G$. Then show that $[G : C(a)] = |Cl(a)|$, where $C(a) = \{g \in G \mid ag = ga\}$ and $Cl(a)$ is the set of all conjugates of $a$. 3

   c) Prove the Cauchy's theorem for finite abelian groups. 3

3. a) State and prove the converse of the Lagrange's theorem for finite abelian groups. 4

   b) Show that every group of order 14 contains exactly six elements of order 7. 3

   c) Show that every group of order 35 is cyclic. 3

4. a) Let $R$ be a ring such that $a^2 = a$ for all $a \in R$. Show that for every $a, b \in R$, $2a = 0$ and $ab = ba$. 3

   b) Let $R$ be a ring with $1(\neq 0)$. Show that $U(R) = \{a \in R \mid ab = 1 = ba \text{ for some } b \in R\}$ is a group under multiplication. 3

   c) Show that the field $\mathbb{Q}$ of all rationals has no proper subfields. Give an another example of such field. 4

5. a) Show that $I = \left\{a + b\sqrt{3} \in \mathbb{Z}\left[\sqrt{3}\right] \big/ a - b \text{ is an even integer}\right\}$ is an ideal in the ring $\mathbb{Z}\left[\sqrt{3}\right]$. 2

   b) Show that the ring $M_{2\times 2}(\mathbb{R})$ of all $2 \times 2$ real matrices is a simple ring. 3

   c) Show that the ring $2\mathbb{Z}$ is not isomorphic to $5\mathbb{Z}$. 3

   d) Determine all ring homomorphisms from $\mathbb{Z}_5$ into $\mathbb{Z}_5$. 2

6. a) Find all prime and maximal ideals in the ring $\mathbb{Z}_8$. 3

   b) Let $F$ be a field. Show that $F[x]$ is an Euclidean domain. 3

   c) Let $R$ be a PID, and $a, b \in R$ be two non-zero elements. Show that $\gcd(a, b)$ exists in $R$ and there are $x, y \in R$ such that $\gcd(a, b) = ax + by$. 4

---