Five Year Integrated M.Sc. Examination, 2019

Semester - VI

Course: MT-3-6-1

Mathematics: Analysis-II

Time: Four Hours **Full Marks: 80**

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings.

Answer ***any eight*** questions.

1. a) Define a topological space. Show that the intersection of any family of topologies for X is a topology for X. 4

   b) Let X be an infinite set and $\tau = \{\phi\} \cup \{G : G \subset X \text{ and } G^C \text{ is finite}\}$. Show that τ is a topology on X. 6

2. a) Let (X, τ) be a topological space and A be a subset of X. Then prove that A is τ open if and only if A contains a τ neighbourhood of each of its points. 4

   b) Let X be a nonempty set and $\mathfrak{F}$ be a family of some subsets of X satisfying the properties:

   i) $X \in \mathfrak{F}$

   ii) $\phi \in \mathfrak{F}$

   iii) if $F_\alpha \in \mathfrak{F},\ \forall \alpha \in \Lambda,$ where $\Lambda,$ is an arbitrary index set, then $\bigcap_{\alpha \in \Lambda} F_\alpha \in \mathfrak{F}$

   iv) $F_1, F_2 \in \mathfrak{F}$ implies $F_1 \cup F_2 \in \mathfrak{F}$.

   Then there exists a unique topology τ for X such that the τ-closed sets of X are precisely the members of $\mathfrak{F}$. 6

3. a) Let A and B be two subsets of a topological space (X, τ). Then show that

   $D(A \cup B) = D(A) \cup D(B).$ 5

   b) Let (X, I) be the indiscrete topological space and A be any subset of X. Find the derived set of A. 5

4. a) Let X be a nonempty set and $P(X)$ be its power set. Let $C : P(X) \to P(X)$ be an operator such that

   i) $C(\phi) = \phi$

   ii) $A \subset C(A)$ for every $A \subset X$

   iii) $C(A \cup B) = C(A) \cup C(B)$ for every $A, B \subset X$

   iv) $C(C(A)) = C(A) \quad \forall A \subset X.$

   Then show that there exists a unique topology τ on X such that for each $A \subset X$, $C(A)$ coincides with τ closure of A. 6

   b) Define a basis for a topology. Find a basis of the usual topology $\mathfrak{U}$ for R, the set of reals. 4

5. a) Let $\mathfrak{B}$ be a family of subsets of a nonempty set X. Then $\mathfrak{B}$ is a base for some unique topology on X if and only if it satisfies the following two properties:

   i) $X = \cup\{B : B \in \mathfrak{B}\}$

   ii) For any $B_1, B_2 \in \mathfrak{B}$ and every point $x \in B_1 \cap B_2$ there exist $B_x \in \mathfrak{B}$ such that

   $x \in B_x \subset B_1 \cap B_2.$ 7

P.T.O.

b) Let $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\} X\}$. Find a local base at each point *a*, *b*, *c*. 3

6. a) Define first countable space. Give an example of it. 3
   b) Show that every second countable space is first countable. Show by an example converse need not be true. 7

7. a) Prove that in a second countable space every open cover of a subset is reducible to a countable subcover. 6
   b) Let (X, τ) be a topological space and let $A \subset X$. Then the family $\tau_A = \{A \cap G : G \in \tau\}$ is a topology on *A*. 4

8. a) A topological space (X, τ) is connected if and only if the only nonempty subset of *X* which is both τ open and τ closed is *X* itself – prove it. 5
   b) It the sets *C* and *D* are connected sets which are not separated, then $C \cup D$ is connected – prove it. 5

9. a) Let (X, τ) be a topological space. Then show that each point of *x* is contained exactly in one component of *X*. 4
   b) Prove that a subset *E* of *R* is connected if and only if it is an interval. 6

10. a) Show that every T_3 space is T_2 and converse is not true. 5
    b) When a topological space is called completely regular? Show that a completely regular space is regular. 5

11. a) Show that a compact T_2 space is T_4. 5
    b) Show that a space *X* is compact if and only if each family of closed set with finite intersection property has a nonempty intersection. 5