M.A./M.Sc. Examination 2018

Semester - I

Mathematics

Course: MMC-11 (New & Old)

(Real Analysis)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***Question No. 6*** and ***any three*** from the rest.

1. a) Show that the following sets are measurable and find their measures:
   i) $\{\sin x : 0 < x < \pi\}$
   ii) $Q \cup \{x \in [0, \infty) : e^x + \sin x > 0\}$
   iii) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{\frac{1}{x} : x \in (1, \infty)\right\}$ 2+2+2

   b) Show that a set with outer measure zero is measurable. 2

   c) Let E be a measurable set. Show that there is a measure zero set F such that $E \cup F$ is a G_δ-set. 4

2. a) Prove that there exists a non-measurable set. 4

   b) State and prove Luzin theorem. 6

   c) If f' exists on $[a, b]$ then show that f' is measurable. 2

3. a) If a sequence $\{f_n\}$ converges in measure to f on a measurable set E then show that there is a subsequence $\{f_{n_k}\}$ which converges almost everywhere to f on E. 5

   b) Define a simple function. Is it measurable? Support your answer.
   If $f : E \to [0, \infty)$ be a measurable function then show that there exists a nondecreasing sequence $\{f_n\}$ of simple function converging to f on E. 1+6

4. a) If f is a bonded and measurable function defined on E then show that f is L-integrable. 3

   b) Let $\{E_n\}$ be a sequence of disjoint measurable sets with $UE_n = E$. If f is Lebesgue integrable on E then show that $\int_{UE_n} f d\mu = \sum_{n=1}^{\infty} \int_{E_n} f d\mu.$ 4

   b) State and prove Bounded convergent theorem.
   Does it hold for Reimann integrable functions. Support your answer. (1+3)+1

P.T.O.

5. a) Let $f: S \to [0,\infty]$ be measurable. Define for $M > 0$.

$$f_M(x) = \min\{f(x), M\}, \quad x \in S \cap [-M, M]$$
$$= \quad 0 \qquad\qquad , \quad x \in S - [-M, M]$$

Then show that 4+1

$$\int_S f\, d\mu = \lim_{M \to \infty} \int_S f_M\, d\mu.$$

And hence find the value of $\int_0^\infty e^{-x} dx$.

b) If f and g are non-negative measurable functions on S, then show that

i) for any $\alpha > 0$, $\int_S (\alpha f)\, d\mu = \alpha \int_S f\, d\mu$ 2+4

ii) $\int_S (f+g)\, d\mu = \int_S f\, d\mu + \int_S g\, d\mu$

c) State Fatou's Lemma. 1

6. Answer ***any two*** of the following: 2×2

a) Find $\int_0^1 f(x) dx$ where

$$f(x) = n \text{ if } x = \tfrac{1}{n}, n = 1, 2, \ldots$$
$$= x \text{ if } x \text{ is rationals in } [0,1] \text{ other than } \left\{\tfrac{1}{n}\right\}$$
$$= x^2 \text{ if } x \text{ is rationals in } [0,1].$$

b) State Dominated convergence theorem. Show that Bounded convergence theorem can be deduced from it.

c) Give examples of F_σ and G_δ sets which are not open or closed.

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