B.Sc.(Honours) Examination, 2019

Semester-IV (CBCS)
Physics (Honours)
Core Course: CC-8 (Theory)
(Mathematical Physics-III)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Answer ***Question No. 1*** and ***any three*** from the rest of the questions.

1. (a) Find the roots of the equation $z^6 + 1 = 0$, where z is a complex number.

(b) Consider the following complex function $f(z) = \begin{cases} z^2; & z \neq i \\ 0; & z = i \end{cases}$. Is the function is continuous or not at $z = i$? State the logic in support of your answer.

(c) If $f(k)$ is the Fourier transform of a function $f(x)$, i.e., $F[f(x)] = f(k)$ then show that $F[f(ax)] = \frac{1}{a} f\left(\frac{k}{a}\right)$ where a is a constant.

(d) Show that Laplace Transform of the function, $f(t) = t\sin(t)$ is $\frac{2as}{(s^2+a^2)^2}$.

(e) If a complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region then show that $\nabla^2 u = \nabla^2 v = 0$ in that region. **5×2=10**

2. (a) Locate the singularities of the function $f(z) = \frac{\sqrt{z}}{z^2+2z+2}$ in a complex plane and mention the nature of the singularities.

(b) Does the following limit exist? Justify.

$$\lim_{z\to 0} \frac{\mathrm{Re}(z^2)+\mathrm{Im}(z^2)}{z^2}$$

(c) Find the radius of convergence of the following series:

(i) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

(ii) $\sum_{n=0}^{\infty} n!\; z^n$

(iii) $\sum_{n=0}^{\infty} (z+5i)^{2n}(n+1)^2$

(iv) $\sum_{n=0}^{\infty} 2^n (z-1)^n$

(d) Can $u(x,y) = 2y - y^2 + x^2 + 2x$ be a real part of an analytic function in a complex plane? **2+2+4+2**

3. (a) Expand $f(z) = \frac{1+z}{1-z}$ about $z = i$ and find the region of validity.

(b) Expand $f(z) = \frac{z+1}{z(z-4)^3}$ about $z = 4$ in the domain $0 < |z-4| < 4$.

(c) Evaluate $\frac{1}{2\pi i}\oint \frac{e^z}{z-2}dz$ over the circles $|z| = 3$ and $|z| = 1$.

(d) Evaluate $\oint \frac{e^{3z}}{z-i\pi}dz$ over the elliptical path described by $|z-2| + |z+2| = 6$.

(e) State and prove Cauchy's theorem. **2+2+2+2+2**

P.T.O.

4. (a) State and prove Jordan's Lemma.

(b) State Cauchy's Residue theorem in a complex plane.

(c) Applying Cauchy's Residue theorem evaluate the integral $\int_0^\infty \sin(x^2)dx$. **4+2+4**

5. (a) Find the Fourier transform of the function $f(x) = \frac{1}{x^2+a^2}$, where a is a positive constant.

(b) Suppose that a particle is represented spatially by the function $f(x) = ae^{-\frac{x^2}{2\sigma^2}}$, where a and σ are positive constants. Show that the half-width of particle profile in the Fourier space is inversely proportional to that in the real space.

(c) Find the Fourier transform of the following function $f(x) = \begin{cases} 1; & |x| < 1 \\ 0; & |x| > 1 \end{cases}$ and hence evaluate $\int_0^\infty \frac{\sin(x)}{x} dx$. 4+2+4

6. (a) If a function $f(x)$ is periodic with period T, then $f(t+T) = f(t)$. Show that the Laplace Transform of such a periodic function is given by $L[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{(1-e^{-sT})}$.

(b) Let us represent Laplace transform of a function $f(t)$ as $L[f(t)] = \bar{f}(s)$ and assume that the first $(n-1)^{th}$ derivatives of $f(t)$ are continuous. Prove that

$L[f^n(t)] = s^n f(s) - s^{n-1}f(0) - s^{n-2}f^1(0) - \ldots\ldots\ldots - f^{n-1}(0)$.

(c) The coordinate (x, y) of a particle moving along a plane curve at any time t are given by $\frac{dy}{dt} + 2x = \sin(2t)$ and $\frac{dx}{dt} - 2y = \cos(2t)$; $t > 0$. With the initial conditions $x(0) = 1$ and $y(0) = 0$, solve the differential equations by using Laplac and Inverse Laplace transform and thus show that the particle moved along the curve: $4x^2 + 4xy + 5y^2 = 4$.

3+2+5

7. Find the Inverse Laplace transform of the following functions:

(a) $\bar{f}(s) = \frac{1}{s(s^2+a^2)}$, where $a$is a constant.

(b) $\bar{f}(s) = \frac{1}{s(s+a)^3}$, where $a$is a constant.

(c) $\bar{f}(s) = \frac{s^2}{(s^2+a^2)^2}$, where $a$is a constant.

(d) $\bar{f}(s) = \frac{1}{(s^2+1)(s^2+9)}$. 2+3+3+2