Use separate answer scripts for each Unit

B.Sc. (Honours) Examination 2019

Semester – IV(CBCS)

Mathematics

Course: CC-8

(Analysis-IV and Differential Equations-II)

Time: Three Hours **Full Marks: 60**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Unit-I (Marks: 3×10=30)

Answer ***any three*** questions

1. a) Prove that the function $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & if\ (x,y)\neq(0,0),\\ 0 & if\ (x,y)=(0,0),\end{cases}$ 3

is continuous at $(0,0)$.

b) Find the directional derivative of $f(x,y)=2x^2-xy+5$ at $(1,1)$ in the direction of unit vector $\overline{\beta}=\left(\frac{3}{5},-\frac{4}{5}\right)$. 3

c) State and prove sufficient conditions for the differentiability for a function of two variables. 4

2. a) If $f(u,v)$ is a differentiable function of two independent variables u and v, which are also differentiable function of one independent variable x, then show that f is a differentiable function of x and $\frac{df}{dx}=\frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{dv}{dx}$. 3

b) State and prove Schwarz theorem. 1+4

c) Approximate the change in the hypotenuse of a right angled triangle whose sides are 6 cm and 8 cm, when the shorter side is lengthened by $\frac{1}{4}$cm and the longer side is shortened by $\frac{1}{8}$ cm. 2

3. a) Let z be a differentiable function in rectangular Cartesian coordinates x and y. If $x=r\cos\theta$ and $y-r\sin\theta$, then prove that $\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$. 4

b) Let $f(0)=0$ and $f'(x)=\frac{1}{1+x^2}$, without using the method of integration, prove that $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$. 3

c) State and prove Euler's theorem for a function of three variables. 3

4. a) Let $f(x,y)=\begin{cases}\dfrac{1}{2}, & if\ y \text{ is rational}\\ x, & if\ y \text{ is irrational}\end{cases}$

and R is the square $[0,1;\ 0,1]$.

Evaluate $\int_0^1 dy\int_0^1 f(x,y)\,dx$ and $\int_0^1 dx\int_0^1 f(x,y)\,dy$ if exist. 3

P.T.O.

4. b) Evaluate $\int_0^1 dy \int_y^1 e^{x^2}\, dx$. 3

c) Using the transformation formula $x+y=u,\ y=uv$, show that $\int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}}\, dy$ transform into $\iint\limits_E e^{\frac{y}{x+y}}\, dx\, dy$, where E is the triangle bounded by $x=0,\ y=0,\ x+y=1$ and whose value is $\dfrac{e-1}{2}$. 4

5. a) Evaluate $\iint\limits_E \dfrac{dx\, dy}{\left(1+x^2+y^2\right)^2}$ over E, which is one loop of the lemniscate $\left(x^2+y^2\right)^2-\left(x^2-y^2\right)=0$. 3

b) Evaluate $\int_{-\infty}^{\infty} e^{-x^2}\, dx.$ 4

c) Evaluate $\iint\limits_R [x+y]\, dxdy$ where R is the rectangle $[0,2;\ 0,1]$, $[a]$ denotes the greatest integer less than or equal to a. 3

Unit-II [Marks: (2×5)+(2×10)=30]

Answer question number 1 and ***any two*** from the rest.

1. Answer ***any five*** questions. 2×5=10

a) Let $y(x)=v_1(x)\cos(3x)+v_2(x)\sin(3x)$ be a solution to $\dfrac{d^2y}{dx^2}+9y=\dfrac{1}{\sin(3x)}$. Find the value of $v_2(x)$.

b) Consider the boundary value problem (BVP) $\dfrac{d^2y}{dx^2}+\alpha\, y(x)=0,\ \alpha\in\mathbb{R},$ with the boundary conditions $y(0)=0,\ y(\pi)=k$ (k is a non-zero real number).
For $\alpha=-1,\ k>0,$ the BVP has a solution $y(x)$ such that $y(x)>0$ for all $x\in(0,\pi)$ – Justify the statement.

c) Given the solution $y_1(x)=(x-1)^{-1}$, find the second solution of the differential equation (DE) $\left(x^2-x\right)\dfrac{d^2y}{dx^2}+(3x-1)\dfrac{dy}{dx}+y=0,\ x\neq 0,1$.

d) Use the substitution $x=\dfrac{1}{t}$ to show that for the DE $\dfrac{d^2y}{dx^2}+\dfrac{1}{2}\left(\dfrac{1}{x^2}+\dfrac{1}{x}\right)\dfrac{dy}{dx}+\dfrac{1}{2x^3}y=0,$ the point $x=\infty\,(t=0)$ is a regular singular point.

e) Give geometrical interpretation of the integrable equation $P\,dx+Q\,dy+R\,dz=0$.

f) Consider the system of linear differential equations (Des)

$$\frac{dx_1}{dt} = 5x_1 - 2x_2,$$

$$\frac{dx_2}{dt} = 4x_1 - x_2,$$

with the initial conditions $x_1(0) = 0,\ x_2(0) = 1.$

Find the value of $\log_e |x_2(2) - x_1(2)|.$

g) Convert the initial value problem (IVP)

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0,$$

$$y(x=0) = 1,$$

$$\left.\frac{dy}{dx}\right]_{x=0} = 3$$

into an IVP for a system in normal form.

h) Consider the ordinary differential equation (ODE)

$$4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$

Find the sum of the roots of the indicial equation of the Frobenius series solution for the above ODE in a neighbourhood of $x = 0$.

2. a) Find at least the first four non-zero terms in a power series expansion about $x = 0$ of a general solution to the given ODE

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = \cos x. \qquad 6$$

b) Solve the simultaneous equations:

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}. \qquad 4$$

3. a) Let the function $p_1(x)$ be differentiable in some interval I of $\mathbb{R}$. Show that the substitution $y(x) = z(x)\exp\left(-\frac{1}{2}\int^x p_2(t)\,dt\right)$ transforms the equation

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = r(x) \quad \text{to}$$

$$\frac{d^2z}{dx^2} + \left[p_2(x) - \frac{1}{2}\frac{dp_1}{dx} - \frac{1}{4}p_1^2(x)\right]z = r(x)\exp\left(\frac{1}{2}\int^x p_1(t)\,dt\right). \qquad 5$$

b) Show that the condition of integrability is satisfied by the equation

$$(1+yz)\,dx + x(z-x)\,dy - (1+xy)\,dz = 0 \qquad 1+4=5$$

and hence solve the equatons.

4. a) State the Picard-Lindelöf's existence and uniqueness theorem on the solution of the IVP

$$\frac{dy}{dx} = f(x, y),$$
$$y(x = x_0) = y_0.$$

Hence, by constructing the approximate solutions, show that $y(x) = \frac{5}{2}e^{2x} - \frac{3}{2}$ is a solution of the IVP

$$\frac{dy}{dx} = 2y + 3,$$
$$y(x = 0) = 1.$$

1+4=5

b) By matrix / vector method, find the general solution of the homogeneous system:

$$\frac{dX}{dt} = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} X, \quad \text{where } X = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Comment on the particular solution if $X(t = 0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. 4+1=5
