M.A./M.Sc. Examination, 2019
Semester-IV
Mathematics
Optional Course: MMO-41(P09) (New Syllabus)
[Rings and Modules-II(Structure of Rings and Modules)]

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions

1. a) If M and N are R–modules, prove that M cogenerates N if and only if there is a subset $H \subseteq Hom_R(N, M)$ such that $\bigcap_H Ker\, f = (0)$. 3
   b) Let M be a right R-module. Prove that M is Artinian if and only if every factor module of M is finitely cogenerated. 4
   c) Give an example of a ring which is right Artinian but not left Artinian. 3

2. a) Let $O \to M_1 \to M \to M_2 \to O$ be a short exact sequence. Prove that M is Noetherian if and only if both M_1 and M_2 are Noetherian. 4
   b) Let $\varphi : M \to M$ be an endomorphism. If M is Noetherian then show that there is $n_0 \in \mathbb{N}$ such that $lm\ \varphi^n \cap Ker\, \varphi^n = (0)$ for all $n \geq n_0$. 3
   c) Let R be a right Artinian ring and $a, b \in R$ be such that $ab = 1$. Show that $ba = 1$. 3

3. a) Let R be a ring with more than one element. If R is right Artinian and R has no divisors of zero then prove that R is a division ring. 3
   b) Prove that an Artinian ring can have only finitely many maximal ideals. 3
   c) Let M be both an Artinian and Noetherian right R-module. If N is a submodule of M, then show that $l(M) = l\left(M/N\right) + l(N)$. 4

4. a) Let R be a PID and M be an R-module. If N, P and Q are submodules of M such that $N = P \oplus Q$, then show that $O(N) = lcm\big(O(P), O(Q)\big)$. 2
   b) Let M be a finitely generated module over a PID R. Prove that $M \simeq Tor(M) \oplus Free(M)$. 4
   c) Let M be a finitely generated torsion module over a PID R. If $O(M) = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$, where $p_1, p_2, \ldots, p_k \in R$ are distinct primes then prove that
   $M = M_{p_1} \oplus M_{p_2} \oplus \ldots \oplus M_{p_k}$, where $M_{p_i} = \{m \in M \mid p_i^{n_i} m = 0\}$ for all $i = 1, 2, \ldots, k$. 4

5. a) Prove that every finite abelian group is a direct sum of cyclic groups. 4
   b) Define invariant factors of a finite abelian group. Prove that two finite abelian groups G and H are isomorphic if and only if they have the same invariant factors. 4
   c) Find invariant factors and elementary divisors of $G = \mathbb{Z}_{50} \oplus \mathbb{Z}_{40} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{24}$. 2

6. a) Define subdirect product of rings. Show that the ring of integers $\mathbb{Z}$ is a subdirect product of the fields $\{\mathbb{Z}_{p_\alpha} \mid \alpha \in \Delta\}$, where $\{p_\alpha \mid \alpha \in \Delta\}$ is the set of all primes in $\mathbb{Z}$. 3
   b) Show that every ring is a subdirect product of sibdirectly irreducible rings. 3
   c) Prove that every Boolean ring is isomorphic to a ring of sets. 4