M.A./M.Sc. Examination, 2019
Semester-IV
Mathematics
Optional Course: MMO-41(P03) (New Syllabus)
(Advanced Real Analysis-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Answer ***Question No. 6*** and ***any three*** from the rest.

1. a) Show that an additive set function on an algebra is nondecreasing if and only if it is nonnegative. 3
   b) When is a sequence of sets called convergent? Prove that a monotone increasing sequence of sets is convergent and converges to its supremum. 5
   c) Define a σ-algebra? Let X be a nonempty set and $\mathfrak{F} = \{A \subset X : A$ is finite or A^C is finite$\}$. Show that $\mathfrak{F}$ is an algebra but not a σ-algebra. 4

2. a) When is a measure space (X, M, μ) said to be complete? Show that there is a complete measure space $(\bar{X}, \bar{M}, \bar{\mu})$ such that $M \subset \bar{M}$ and $\bar{\mu}$ is extension of μ on $\bar{M}$. 6
   b) Show that there is an outer measure on $\mathbb{R}^2$, constructed from premeasure such that open sets need not be measurable. 4
   c) Let ψ be an additive set function on a algebra $\mathcal{A}$. Let E_1 and E_2 be members of $\mathcal{A}$ with $E_1 \subset E_2$ and also let $\psi(E_2)$ be finite. Show that $\psi(E_2 - E_1) = \psi(E_2) - \psi(E_1)$. 2

3. a) State and prove Caratheodary Hahn extension theorem w.r.t outer measure. 8
   b) Give definition of Lebesgue-Stieltjes outer measure μ_f^*. Show that
   $$\mu_f^*(a, b] = f(b) - f(a).$$ 4

4. a) Suppose f be a measurable function defined on a measure space (X, M, μ). Let $E \subset X$ be a measurable set. Give definition of $\int_E f d\mu$. 3
   b) Suppose X is a metric space and $\phi : X \to X$ is continuous and f is measurable w.r.t a metric measure. Show that $\phi \circ f$ is measurable. 3
   c) If f is integrable on a measure space (X, M, μ) and c is constant, then show that cf is integrable and $\int_X cf\, d\mu = c\int_X f\, d\mu$. 6

5. a) If $\mu(E) = 0$ for a measurable set E then show that $\int_E f\, d\mu = 0$ for any measurable function f. 3
   b) Let f be integrable on a measurable set E and $f = g$ a.e on E. Show that
   $$\int_E f\, d\mu = \int_E g\, d\mu.$$ 3
   c) Let f be nonnegative measurable function defined on a measurable set E. If $\alpha > 0$, then show that
   $$\mu\{x \in E : f(x) > \alpha\} \le \frac{1}{\alpha}\int_E f\, d\mu.$$ 3

P.T.O.

d) Let $f(x) = 0$ if $x < 0$

$= 1$ if $0 \le x < 1$

$= 2 \quad x \ge 1$

and μ_f be the associated Lebesgue-Stieltjes measure. Show that

$\mu_f(0, 1) < \mu_f(0\ 1] < \mu_f[0, 1]$. 3

6. Answer ***any two***: 2x2=4

a) Show that greatest integer function [x] defined on any finite interval is a simple function.

b) Let $X = \mathbb{N}, \mathbb{N}$ being the set of natural numbers. If $\mathcal{A} = \mathcal{P}(\mathbb{N}), A_n = \{n, n+1, ...\}$ and $\mu(A) =$ number of elements in A, then show that

$\lim_{n\to\infty} \mu(A_n) \ne \mu\left(\lim_{n\to\infty} A_n\right)$.

c) Show that Lebesgue outer measure is regular.

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