M.A./M.Sc. Examination, 2019
Semester-IV
Mathematics
Optional Course: MMO-41(P02) (New Syllabus)
(Advanced Functional Analysis-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer any ***four*** questions.

1. a) Prove that for any nonempty subset M of a Hilbert space H, the span of M is dense in H iff $M^{\perp} = \{\theta\}$ where $M^{\perp}$ denotes the orthogonal complement of M. 3
   b) Show that every orthonormal subset in a separable Hilbert space is countable. 3
   c) Prove that every closed convex subset of a Hilbert space H has a unique member of smallest norm. 4

2. a) Prove that for every bounded linear operator $T : H \to H$, where H is a Hilbert space, its adjoint operator T^* exists with $\|T\| = \|T^*\|$. 4
   b) Let T_1 and T_2 be two bounded linear operators defined on a Hilbert space H to itself. Then prove the following:
   (i) $\left(T_1^* + T_2^*\right) = \left(T_1 + T_2\right)^*$ (ii) $\|T_1\, T_1^*\| = \|T_1^*\, T_1\| = \|T_1\|^2$. 4
   c) Let $T : H \to H$ be a continuous linear operator, where H is a Hilbert space. Show that T can be expressed uniquely as $A + iB$ where A and B are self-adjoint operators. 2

3. a) If T_1 and T_2 are normal operators defined on a Hilbert space H such that either commutes with the adjoint of the other. Prove that $T_1 + T_2$ and $T_1 T_2$ are normal operators on H. 4
   b) When is a bounded linear operator defined on a Hilbert space into itself is said to be positive? Suppose T_1 and T_2 are two positive operators defined on a Hilbert space into itself, show that $T_1 T_2$ is also positive if $T_1\, T_2 = T_2\, T_1$. 1+5

4. a) Define a compact linear operator. Let $T : X \to Y$ be a compact operator, where X and Y are normed linear spaces and $\{x_n\}$ be a weakly convergent sequence in x say $x_n \xrightarrow{\omega} x$. Prove that $\{Tx_n\}$ is strongly convergent in Y and has the limit $y = Tx$. 1+3
   b) Show that the spectrum $\sigma(T)$ of a bounded linear operator T on a complex Banach space X is closed. 4
   c) Prove that every self-adjoint operator is normal but converse may not be true in general. 2

5. a) Define spectral radius $r_\sigma(T)$ of a bounded linear operator T defined on a complex Banach space X and show that $r_\sigma(T) = \overline{\lim_{n\to\infty}} \sqrt[n]{\|T^n\|}$. 1+4
   b) Let $T : H \to H$ be a self-adjoint operator, where H is a Hilbert space. Show that spectrum $\sigma(T)$ lies in the closed interval [m, M] where $m = \underset{\|x\|=1}{Inf} \langle Tx, x\rangle$ and $M = \underset{\|x\|=1}{Sup} \langle Tx, x\rangle$. 3
   c) Let $T \in B(H, H)$ where H is a Hilbert space. Prove that $\sigma(T^*) = \{k \in \mathbb{C} : \overline{k} \in \sigma(T)\}$. 2

P.T.O.

6. a) Let A be a complex Banach algebra with identity. Show that for any $x \in A$, the spectrum $\sigma(x) \neq \varphi$. 5

   b) Let P_1 and P_2 be two projection operators on a Hilbert space H. Prove that $P_1 + P_2$ is a projection operator on H iff $P_1(H)$ and $P_2(H)$ are orthogonal. 3

   c) Show that every unitary operator on a Hilbert space is isometric but not conversely. 2