M.A./M.Sc. Examination 2019

Semester - IV

Mathematics

Optional Course: MMO-41(P01) (New Syllabus)

(Advanced Complex Analysis-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings.

*Answer **any four** questions.*

1. a) Let $f(z)$ be meromorphic in $|z|<\infty$ and $a_1, a_2, \ldots$ be the poles of $f(z)$ where $0<|a_1|\le|a_2|\le|a_3|\le\ldots$. Suppose that there is a sequence of closed contours C_n, such that C_n includes $a_1, a_2, \ldots, a_n$ but no other poles and R_n is the minimum distance of C_n from the origin, which tends to infinity with n, while L_n, the length of C_n is $O(R_n)$. Now if $f(z)=0(R_n)$ on C_n and residue at a_m be b_m, then show that

$$f(z)=f(0)+\sum_{n=1}^{\infty} b_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right),$$

for all values of z except the poles. 5

b) Let $n(x)$ denote the number of zeros of f in $|z|\le x$, (zeros are counted with multiplicities). Suppose that f is analytic in $|z|<R_1$ and $f(0)\ne 0$. Let $r_1, r_2, r_3, \ldots$ be the moduli of zeros of f in $|z|<R_1$, arranged in the non-decreasing order. If $r_m \le R < r_{m+1}$, then show that

$$\int_0^R \frac{n(x)}{x}dx=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\mathrm{R}e^{i\phi}\right)\right|d\varphi-\log|f(0)|. \quad 5$$

2. a) State and prove Poisson-Jensen formula. 5

b) Define the Nevanlinna's characteristic function $T(r, f)$. Find $T(r, f)$ when $f(z)=e^{z^2}$. 3

c) If $f_1, f_2, \ldots, f_p$ be p meromorphic function in $|z|<r$, then show that

(i) $m\left(r, \prod_{k=1}^{p} f_k\right)\le\sum_{k=1}^{p} m(r, f_k)$; (ii) $N\left(r, \prod_{k=1}^{p} f_k\right)\le\sum_{k=1}^{p} N(r, f_k)$. 2

3. a) Show that $|T(r, f+a)-T(r, f)|\le\log^+|a|+\log 2$, where '$a$' $(\ne\infty)$ is a complex constant. 2

b) State and prove Nevanlinna's first fundamental theorem. 3

c) If $f(z)$ is meromorphic in $|z|<R$ and $g(z)=\dfrac{af(z)+b}{cf(z)+d}$, where *a, b, c, d* are constants such that $ad-bc\ne 0$ and if $f(0)\ne\infty$, $g(0)\ne\infty$, then show that

$$T(r, g)=T(r, f)+O(1)\ (0<r<R). \quad 3$$

d) For any 'a', show that $N(r, a)$ is an increasing convex function of $\log r$. 2

4. a) State and prove Nevanlinna's second fundamental theorem. 4

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b) If $f(z)$ is regular in $|z| \le R$, then show that

$$T(r, f) \le \log^+ M(r, f) \le \frac{2R}{R-r} T(R, f), \ \ 0 \le r < R.$$ 3

c) For the function $f(z) = e^{e^z}$, show that $\dfrac{T(r, f)}{\log M(r, f)} \to 0$ as $r \to \infty$. 3

5. a) Define the order of a meromorphic function $f(z)$. Find the order of $e^{z/2}$. 3

b) Suppose that $f(z)$ is a non-constant meromorphic function in the complex plane and $a_1, a_2, ..., a_q \, (q \ge 3)$ are q distinct values in the extended complex plane then prove that

$(q-2)T(r, f) \le \sum_{k=1}^{q} \bar{N}(r, a_k) + S(r, f)$, where $S(r, f)$ is defined by the second fundamental theorem. 3

c) State Nevanlinna's theorem on deficient functions. 2

d) Define $\delta(a, f)$ and find $\delta(o, e^{2z})$. 2

6. a) State and prove Milloux's theorem. 6

b) If two non-constant rational functions f and g share two distinct values a, b *CM* and if there exists a point α such that $f(\alpha) = g(\alpha)$ distinct from a and b, then show that $f \equiv g$. 2

c) Define Weierstrass elliptic function $\mathcal{P}(z)$. Show that $\mathcal{P}(z)$ is an even function. 2