M.A./M.Sc. Examination 2019

Semester - IV

Mathematics

Course: MMC-41(New Syllabus)

(Differential Geometry & Manifold Theory)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
*Answer **any four** questions.*

1. Define normal section of a smooth surface M at a point $P\left(\left(x^1(u,v), x^2(u,v), x^3(u,v)\right)\right)$ in the direction $\vec{w}$. Hence find the normal section to the surface $M=\left\{\left(x^1, x^2, x^3\right):\left(x^1\right)^2+\left(x^2\right)^2=1, -2\le x^3\le 2\right\}$ at the point (0, 1, 0) in the direction $\vec{w}_1=(0,0,1), \vec{w}_2=(0,1,0)$ and $\vec{w}_3=\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. Find curvature and torsion to these curves. 1+6+3

OR

Define manifold $\mathcal{M}$ of dimension m and a affine linear connection on $\mathcal{M}$. Define covariant derivative of a vector field in terms of connection. Derive expressions for $\nabla_X f, \nabla_X Y$ and $\nabla_X \omega$ for $f:\mathcal{M}\to\mathbb{R}, X=X^i(\tilde{x})\frac{\partial}{\partial x^i}, Y=Y^i(\tilde{x})\frac{\partial}{\partial x^i}, \omega=\omega_i(\tilde{x})dx^i$. 10

2. Define the first fundamental form of a smooth surface $M=\left\{\left(x^1(u,v), x^2(u,v), x^3(u,v)\right):(u,v)\in\Omega\sqsubset\mathbb{R}^2\right\}$. Give geometrical interpretation of coefficients involved in the first fundamental form. Find the first fundamental form for the surface $M=\{\sin u\cos v, \sin u\sin v, \cos u\}$ $(0\le u\le\pi, 0\le v\le 2\pi)$. 2+1+7

OR

Define parallel displacement of a vector on a curved space. If K_ν is the parallel displacement of A_ν at $P(\tilde{x})$ to a neighbouring point $Q(\tilde{x}+d\tilde{x})$, prove that

$dA_\nu=K_\nu-A_\nu=A^\mu Y_\mu^n Y_{n,\nu,\sigma}\,dx^\sigma=A^\mu\Gamma_{\mu\nu\sigma}dx^\sigma$. Where $\Gamma_{\mu\nu\sigma}$ is Christoffel symbol of first kind.

Derive the relation between Christoffel symbol of first, second kind and the metric tensor $g_{\mu\nu}$ of the curved space. 1+5+2+2

3. Explain the symbols g_{pq} and Γ^p_{qr} involved in the derivatives of position vector of a point on a curve on the smooth surface with respect to the parameter involved. Define the geodesic of a surface. Use this definition to derive the equation for geodesic on a surface. Hence show that geodesics on a sphere are great circles on it. 2+2+1+2+3

OR

Define Lie group and Lie group of transformations. Verify whether $S=\left\{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}, a\in\mathbb{R}\right\}$ and $T_{(a,b)}:\mathbb{R}\to\mathbb{R}$ given by $x^*=T_{(a,b)}x=ax+b$ are Lie group and Lie group of transformations.

2+6+2

P.T.O.

4. Find g_{pq} and Γ^p_{qr} for the surfaces $M_1 = \left\{\left(x^1, x^2, x^3\right) : \left(x^1\right)^2 + \left(x^3\right)^2 = 1, -\infty \le x^2 \le \infty\right\}$ and $M_2 = \left\{x_1^2 + x_2^2 + x_3^2 = 1, x_1, x_2, x_3 \in \mathbb{R}\right\}$. 5+5

OR

What do you mean by a vector field $Y = Y^i\left(\tilde{x}\right)\frac{\partial}{\partial x^i}$ on a manifold $\mathcal{M}$ is parallel along a Curve C on $\mathcal{M}$? Define geodesic on a manifold and derive its equation. Find geodesic on a 2-sphere. 2+3+5

5. Define second fundamental form of a surface. Compute the second fundamental form of the surface of revolution $\vec{X}(u, v) = \left(f(u)\cos v, f(u)\sin v, g(u)\right)$. 2+8

OR

a) Define k-form on a manifold $\mathcal{M}$ of dimension m. If $\mathcal{M}$ be a three dimensional manifold with a local coordinate (x, y, z) around a point $P \in \mathcal{M}$, find the basis of $\Lambda^{(k)}$, $k = 1, 2, 3, 4$. Define wedge product and state fundamental properties of this operation. If $\omega \in \Lambda^{(m-1)}$ and $\overline{\omega} \in \Lambda^{(2)}$ are two forms on $\mathcal{M}$, find $\omega \wedge \overline{\omega}$. 1+2+2

b) Define exterior derivative of a k-form on a manifold $\mathcal{M}$ of dimension $m(m \ge k)$. State Poincare lemma and prove it for 0-form $f : \mathcal{M} \to \mathbb{R}$. State its correspondence with operations gradient, divergence and curl, if any. 1+2+2

6. State Gauss curvature involving first and second fundamental forms. Find the parameter dependence of Gauss curvature at a point (u, v) on the surface $\left(u, v, f(u, v)\right)$. 2+8

OR

What do you mean by commutator? State few important properties of this operation. Define tangent bundle and a distribution on a manifold. When a distribution is said to be integrable and in involution. Verify which of the following distributions $\mathcal{D}_1 = \left\{(1, 0, y), (0, 1, -x)\right\}$ and $\mathcal{D}_2 = \left\{(1, 0, x), (0, 1, y)\right\}$ are integrable? 1+2+2+2+3

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