M.A./M.Sc. Examination, 2019

Semester - II

Mathematics

Core Course: MMC-24 (Old Syllabus)

(Partial Differential Equations)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

(Notations and symbols have their usual meanings unless otherwise stated)

Answer ***any four*** questions.

1. a) Give an idea about the characteristic strip for a first order nonlinear PDE of the form
$$f\left(x,\, y,\, u,\, u_x,\, u_y\right)=0,\; u=u\left(x,\, y\right).$$
Hence, solve the Cauchy problem $pq=xy,\; u\left(x,\, 0\right)=x.$ 1+5

   b) Solve the following Cauchy-Euler type PDE:
$$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=x^m y^n, \text{ where } \left(m+n\right)\neq 0,1.$$ 4

2. a) Reduce $u_{xx}+xu_{yy}=0,\; x>0$ to canonical form. 5

   b) Solve the following PDE by using Monge's method:
$$y^2u_{xx}+2xyu_{xy}+x^2u_{yy}+xu_x+yu_y=0.$$ 5

3. a) What do you mean by Dirichlet type boundary conditions for a PDE? Prove that the solutions of the Dirichlet problem depend continuously on the boundary data. 2+3

   b) Obtain the solution of the free vibrations of a semi-infinite string governed by

   PDE: $u_{tt}=C^2u_{xx},\; 0<x<\infty,\; t>0$

   IC's: $u\left(x,\, 0\right)=f\left(x\right)$

   $u_t\left(x,\, 0\right)=g\left(x\right)$

   $u,\, u_x\to 0$ as $x\to\infty.$ 5

4. a) A semi-infinite solid $x>0$ initially at temperature zero. At time $t=0,$ a constant temperature $u_0>0$ is applied and maintained at the face $x=0.$ Find the temperature at any point of the solid at any time $t>0.$ 5

   b) Solve the following boundary value problem in the half-phase $y>0,$ described by

   PDE: $\nabla^2u=0,\; -\infty<x<\infty,\; y>0$

   BC: $u\left(x,\, 0\right)=f\left(x\right),\; -\infty<x<\infty$

   $u\left(x,\, y\right)$ is bounded as $y\to\infty;\; u$ and u_x both vanish as $\left|x\right|\to\infty,\; \nabla^2\equiv\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}.$ 5

5. a) Find a surface passing through the parabolas $y^2+u^2=4ax,\; u=0$ and $y^2+u^2=-4ax+1,$ $u=1$ among the family of surfaces represented by the PDE $xu_{xx}+2u_x=0.$ 4

   b) Using the method of separation of variables, solve the following PDE:

   PDE: $u_t=u_{xx},\; 0<x<2019,\; t>0,$

   IC: $u\left(x,\, 0\right)=\begin{cases} x, & 0\leq x\leq 1, \\ 2019-x, & 1\leq x\leq 2019, \end{cases}$

   BC's: $u_x\left(0,\, t\right)=u_x\left(2019,\, t\right)=0.$ 6

P.T.O.

6. a) Show that if ϕ is a harmonic function in $\mathbb{R}$ and $\frac{\partial \phi}{\partial n} = 0$ on $\partial\mathbb{R}$, then ϕ is a constant in $\overline{\mathbb{R}}\left(= \mathbb{R} \cup \partial\mathbb{R}\right)$. 2

b) Show that the solution $u = u\left(\gamma, \theta\right)$ to the exterior boundary value problem

$$\nabla^2 u = 0,\ \gamma > R,\ 0 \le \theta < 2\pi,$$

$$u\left(R, \theta\right) = f\left(\theta\right),\ 0 \le \theta < 2\pi,$$

is given by Poisson's integral formula

$$u\left(\gamma, \theta\right) = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} \frac{\left(\gamma^2 - R^2\right) f\left(\varphi\right) d\varphi}{R^2 + \gamma^2 - 2R\gamma \cos\left(\theta - \varphi\right)}.$$ 8