M.A./M.Sc. Examination, 2019

Semester - II

Mathematics

Core Course: MMC-23 (Old Syllabus)

(Algebra-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings

Answer ***any four*** questions

1. a) Let R be an integral domain. Prove that R and $R[x]$ have the same characteristic. 3

   b) Find all units of $\mathbb{Z}[x]$. 2

   c) Let R be a commutative ring with unity 1. If $R[x]$ is a principal ideal domain then show that R is a field. Hence show that $\mathbb{Z}[x]$ is not a principal ideal domain. 4+1

2. a) Let E be a Euclidean domain. Let $a, b, q, r \in E$ be such that $a = bq + r$, where $b \neq 0$ and $r \neq 0$. Show that $\gcd(a, b) = \gcd(b, r)$. 2

   b) In the domain $\mathbb{Z}\left[i\sqrt{5}\right]$, show that $\gcd\left(2, 1+i\sqrt{5}\right) = 1$. 4

   c) In a principal ideal ring R, show that every irreducible element is prime. Does every prime element in a principal ideal ring an irreducible element? Justify your answer. 3+1

3. a) Show that every principal ideal domain is a factorization domain. 4

   b) Show that $f(x) = x^4 - 5x^2 + x + 1$ is irreducible in $\mathbb{Z}[x]$. 4

   c) Give an example of a primitive polynomial which has no root in $\mathbb{Q}$, but is reducible over $\mathbb{Z}$. 2

4. a) Let V be a vector space. Suppose $W \subset V$ is an invariant subspace of a linear operator $T: V \to V$. Prove that in $(T) = \begin{bmatrix} m(T_W) & B \\ O & C \end{bmatrix}$, for matrices B and C of same order. 5

   b) Find the minimal polynomial of the matrix
$$\begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$ 5

5. a) Show that every orthonormal set in an n-dimensional inner product space V can be extended to a basis of V. 5

   b) Let T be linear operator on an inner product space V. Prove that $\|T(x)\| = \|x\|$ if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$. 5

6. a) Let V be a vector space. Determine all possible Jordan canonical forms J for a linear operator $T: V \to V$, whose characteristic and minimal polynomials are given by $(t-2)^4(t-3)^2$ and $(t-2)^3(t-3)^2$. 5

   b) Let V be a n-dimensional vector space and $T: V \to V$ be a linear operator be such that characteristic polynomial splits. Prove that T has a Jordan canonical form. 5