M.A./M.Sc. Examination, 2019

Semester - II

Mathematics

Core Course: MMC-23 (New Syllabus)

(Algebra-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings

Answer ***any four*** questions

1. a) Let R be a ring with unity. Then show that $R[x]/\langle x\rangle \simeq R$. Is $\langle x\rangle$ maximal in $R[x, y]$? Justify your answer. 2+1

   b) In $\mathbb{Z}[i]$, if $a+bi$ is an element such that a^2+b^2 is a prime integer, then show that $a+bi$ is a prime element. 3

   c) Show by an example that it is possible to find two elements a, b in a Euclidean domain such that $v(a)=v(b)$ but a, b are not associates. 2

   d) Let R be an integral domain. Show that the units of $R[x]$ are contained in R. 2

2. a) In $\mathbb{Z}[i]$, find $\gcd(9-5i, -9+13i)$. 4

   b) Show that the integral domain $\mathbb{Z}\left[\sqrt{10}\right]=\left\{a+b\sqrt{10} : a, b\in\mathbb{Z}\right\}$ is a factorization domain. 3

   c) If the integral domain D satisfies the *ACCP*, show that the polynomial ring $R[x]$ satisfies the *ACCP*. 3

3. a) Let R be a unique factorization domain. Suppose $f(x)$ and $g(x)$ be two nonzero polynomials in $R[x]$. Then show that there exist a unit $u\in R$ such that

   $$\text{cont}\big(f(x)g(x)\big)=u\ \text{cont}\big(f(x)\big)\ \text{cont}\ g(x).$$

   Show that every irreducible polynomial in $\mathbb{Z}[x]$ is primitive. 3+1

   b) Find all irreducible monic quadratic polynomials over $\mathbb{Z}_3$. 3

   c) Show that any irreducible real polynomial is either linear or quadratic. 3

4. a) Let T be the linear operator on $\mathbb{R}^4$ defined by $T(a, b, c, d)=(a, b, 2c, 3d)$. Show that $\mathbb{R}^4$ is a direct sum of eigen spaces of T. 5

   b) Let T be a linear operator on a vector space V. Show that either V is a T-cyclic subspace of itself or $T=\lambda I$ for some scalar λ. 3

   c) Find minimal polynomial of the matrix

   $$\begin{bmatrix} 4 & -1 & 0 & 0 & 0\\ 1 & 2 & 0 & 0 & 0\\ 0 & 0 & 3 & 1 & 0\\ 0 & 0 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$ 2

5. a) Let T be a linear operator on a finite-dimensional vector space V, whose characteristic polynomial splits, and suppose β is a basis for V such that β is a disjoint union of cycles of generalized eigen vectors of T. Then show that β is a Jordan-Canonical basis for V. 5

P.T.O.

b) Let T be a linear operator on a finite-dimensional vector space V, whose characteristic polynomial splits. Suppose that β is canonical basis for T, and let λ be an eigenvalue of T. Show that $\beta \cap G_\lambda$ is a basis for G_λ. 3

c) Suppose A is a complex matrix with only real eigenvalues. Show that A is similar to a matrix with only real entries. 2

6. a) Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all $x \in V$. Show that T is one-to-one. 3

b) Suppose T be a linear operator on the inner product space $P_1(\mathbb{R})$ with an inner product defined by $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt, \forall f, g \in P_1(\mathbb{R})$. If for $f \in P_1(\mathbb{R})$ $T(f) = f + 3f$, then evaluate $T^{*^{-1}}(4-2t)$. 4

c) Suppose V and W be two inner product spaces with the inner product $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively. If $T: V \to W$ be a linear transformation, then show that $\langle T^*(x), y\rangle_1 = \langle x, T^*(y)\rangle_2$ for all $x \in W$ and $y \in V$. 3
