M.A./M.Sc. Examination, 2019

Semester - II

Mathematics

Core Course: MMC-22 (New Syllabus)

(For Regular & Back Candidates)

(Topology)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions

1. a) Give definitions of a maximal element and least upper bound of a partial ordered set. Give an example of a set with least upper bound but having no maximal element. 2

   b) State Axiom of choice and Zorn's Lemma. Prove that axiom of choice implies Zorn's Lemma. 4

   c) If u and v be two cardinal numbers then show that $u \le v$ and $v \le u$ implies $u = v$. 4

2. a) Prove that the intersection of any family of topologies for a nonempty set X is a topology for X. 2

   b) Let (X, τ) be a topological space and A be a subset of X. Then A is τ-open if and only if A contains a τ-nbd of each of its points. 3

   c) Let $(X, \mathcal{D})$ be a discrete topological space and A be any subset of X. Find the derived set of A. 2

   d) Prove that a set F is closed in a topological space (X, τ) if and only if $D(F) \subset F$. 3

3. a) Let X be a nonempty set and $P(X)$ be its power set. Let C be an operator $C : P(X) \to P(X)$ such that

   i) $C(\phi) = \varphi$;

   ii) $A \subset C(A)$, for every $A \subset X$;

   iii) $C(A \cup B) = C(A) \cup C(B)$, for every $A, B \subset X$;

   iv) $C(C(A)) = C(A), \forall A \subset X$.

   Then show that there exists a unique topology τ on X such that for each $A \subset X, C(A)$ coincides with τ-closure of A. 4

   b) Define a base for a topology τ on a nonempty set X. Let $\mathcal{B}$ be a family of subsets of a nonempty set X satisfying the following two properties:

   i) $X = U\{B : B \in \mathcal{B}\}$

   ii) For any $B_1, B_2 \in \mathcal{B}$ and every point $x \in B_1 \cap B_2$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset B_1 \cap B_2$. Then prove that $\mathcal{B}$ is a base for some unique topology on X. 3

   c) Prove that every second countable topological space is Lindelöff. Is the converse true? Justify. 3

4. a) Prove that a continuous image of a Lindelöff space is Lindelöff. 3

   b) Show that every compact subset of a Hausdorff topological space is closed. Show that this statement need not be true in general if the space is not Hausdorff. 5

   c) Show that a metric space is always normal. 2

P.T.O.

5. a) When is a topological space said to be regular? Give an example of a topological space which is regular but not Hausdorff. 4
   b) Define a Tychonoff space. Show that a Tychonoff space is T_3. 3
   c) Show that a space is T_1-space if and only if each one-pointic set is closed. 3

6. a) Let (Y, τ_y) be a subspace of a topological space (X, τ) and let A and B be two subsets of Y. Then A, B are τ-separated if and only if they are τ_y-separated. 2
   b) A topological space (X, τ) is connected if and only if the only nonempty subset of X which is both τ-open and τ-closed is X itself – prove it. 4
   c) Let (X, τ) be a topological space. Then prove the following:
      i) each point in X is contained exactly in one component of X.
      ii) each connected subset of X is contained in some components. 4