M.A./M.Sc. Examination 2019

Semester - II
Mathematics
Core Course: MMC-21 (New & Old Syllabus)
(Functional Analysis)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.
Answer ***any four*** questions.

1. a) When is a subset in a metric space called compact? Show that every compact subset in a metric space is closed and bounded. Does the converse hold? Support your answer. 1+3+1

   b) Let A be a compact subset in a metric space (X, d). Show that for every subset B of X there exists a point $x_0 \in A$ such that $d(x_0, B) = d(A, B)$. 3

   c) If a discrete metric space (X, d) is compact then X contains finitely many points. 2

2. a) When are two norms on a linear space said to be equivalent? Prove that any two norms defined on a finite dimensional linear space are equivalent. 4

   b) Verify that the sequence space l_∞ is a Banach space w.r.t a norm to be defined by you. 4

   c) Examine if the function $\|\cdot\|$ defined on $\mathbb{R}^2$ by $\|x\| = \left(|x_1|^{\frac{1}{2}} + |x_2|^{\frac{1}{2}}\right)^2$, $x = (x_1, x_2) \in \mathbb{R}^2$ is a norm. 2

3. a) Iff the closed unit ball in a normed linear space is compact, then show that X is finite dimensional. 3

   b) When is a linear operator T over a normed linear space said to be bounded? Prove that every linear operator defined on a normed linear space is continuous if and only if it is bounded. 1+3

   c) Consider the normed linear space $C[0, 1]$ w.r.t sup. norm. Let $t_0 \in [0, 1]$ be fixed and define a functional f on $C[0, 1]$ by $f(x) = x(t_0) \forall x \in C[0, 1]$. Prove that f is a bounded linear operator. Also find $\|f\|$. 3

4. a) State and prove Uniform Boundedness Principle theorem. 1+5

   b) Show that every open ball in a normed linear space is a convex set. 2

   c) Examine if $C[a, b]$ w.r.t sup norm is a Hilbert space. 2

5. a) In an inner product space $(X, \langle\ \rangle)$, prove that $|\langle x, y\rangle|^2 \le \|x\|\,\|y\| \;\forall x, y \in X$ where $\|x\|^2 = \langle x, x\rangle$. Also show that X is a normed linear space w.r.t a norm induced by the inner product. 5

   b) Prove that $\mathbb{C}^n$ is an inner product space w.r.t an inner product to be defined by you. 3

   c) Prove that any orthonormal set of vectors in an inner product space are linearly independent. 2

P.T.O.

6. a) Define weak convergence and strong convergence of a sequence of elements in a normed linear space. Show that in a finite dimensional normed linear space notion of weak convergence and strong convergence of a sequence coincides. 5

b) Let $(X, \langle\ \rangle)$ be an inner product space and $\{e_n\}$ be an orthonormal sequence in X. If $\sum \alpha_n e_n$ converges to $x \in X$, then show that $\alpha_n = \langle x, e_n \rangle \forall n$. 2

c) Show that a normed linear space $(X, \|\ \|)$ is a Banach space if and only if $\{x \in X : \|x\| = 1\}$ is complete. 3