Five Year Integrated M.Sc. Examination, 2017

Semester-VIII

Course: MT-4-8-2A (Old)

(Analysis-IV)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Answer ***any four*** questions.

1. a) Define contraction mapping. Show that contraction mapping is continuous. 2+2

   b) Let $X(\neq \phi)$ be a complete metric space and $T: X \to X$ be a contraction on *X*. Then show that *T* has precisely one fixed point. 6

2. a) Define Banach space. Show that l^{∞} is a Banach space. 1+6

   b) If the sequences $\{x_n\}_n$ and $\{y_n\}_n$ in a normed linear space X converges to x and y respectively. Show that $\{x_n + y_n\}_n$ converges to $x + y$. 3

3. a) Let X, Y are normed linear space and $T: X \to Y$ be a linear operator. Then show that *T* is continuous iff *T* is bounded. 4

   b) Define bounded linear functional. 2

   c) Let $X = C[0,1]$ and $f: X \to \mathbb{R}$ defined by $f(x) = \int_0^1 x(t)\,dt,\ \ \forall x = x(t) \in X$. Then show that *f* is a bdd linear functional with $\|f\| = 1$. 4

4. a) Define dual space. If X be a normed linear space, $a \in X$ and $|f(a)| \leq C,\ \forall f \in X'$ with $\|f\| = 1$, then show that $\|a\| \leq C$ where C be any real number and X' is the dual space of *X*. 1+3

   b) Let $(X, \|\ \|)$ *and* $(Y, \|\ \|')$ be two Banach spaces. If $T: (X, \|\ \|) \to (Y, \|\ \|')$ be a closed mapping, then show that *T* is bounded. 6

5. a) Define inner product space. If X be an inner product space, then show that $\forall x, y \in X$, $\|x + y\|^2 + \|x - y\|^2 = \left[\|x\|^2 + \|y\|^2\right]$. 2+2

   b) Let $T: H \to H$ be a bounded self adjoint linear operator on a Hilbert space *H*, then show that $\langle Tx, x\rangle$ is real $\forall x \in H$. 2

   c) Let $T: H \to H$ be an unitary operator on a Hilbert space *H*. Show that $\|T\| = 1$. 2

   d) Give an example of a normal operator which is not unitary. 2

6. a) Define fundamental group. Give an example of a fundamental group. 4

   b) A space X is simply-connected iff $\exists$ a unique homotopy class of paths connecting any two points in *X*. 4

   c) State Euler-Poincare theorem. 2