Five Year Integrated M.Sc. Examination 2017
Semestar-VI
Course: PH-3-6-1
(Advanced Statistical Mechanics)

Time: Four Hours **Full Marks 80**

Answer Question Number 1. and any six fron the rest of the questions. In numerical problems, please write the actual numbers obtained by you.

1. a) Starting from the basic equation of big-bang cosmology, show that the cosmological scale factor ($R(t)$) increases with time ('t', that is approximately estimated from the moment of big-bang) as $R(t) \propto t^{1/2}$ during the expansion of the (black-body) radiation-dominated early universe. (**5 Marks**)

1. b) From question number [1. a)], show that the temperature ($T(t)$) of the early universe varied with time (t) approximately as $T(t) \propto t^{-1/2}$ during an (assumed) quasi-static and adaibatic (or isentropic) expansion of that early universe. Next, show that the time ($t_{\rm o}$), that might had been taken by the early universe to cool down to a tempreature $T \sim 10^{9^{\rm o}}$ K is about a few hundred seconds. The values of the relevant universal constants are given as: G (gravitational constant) = 6.67×10^{-8} dyne cm^2 gm^{-2}, $\sigma_{\rm B}$(Stefan − Boltzmann constant) = 5.67×10^{-5} erg cm^{-2} s^{-1} K^{-4}, c (Speed of light in vacuum) = 3×10^{10} cm s^{-1}. (**3 Marks**)

1. c) Consider the (outdated but simple) $\alpha-\beta-\gamma$ theory of primordial nucleosynthesis. Given that the average of the total cross-section ($\sigma_{\rm total}$) of the reaction: $n+p \rightarrow d+\gamma$(photon) multiplied by the relative velocity (v) between the neutrons and the protons at a temperature $T \sim 10^{9^{\rm o}}$ K is $< \sigma_{\rm total} v > \sim 10^{-20}$ cm^3 s^{-1}, approximately determine the number density $n(t_{\rm o})$ of neutrons and protons at the time '$t_{\rm o}$' estimated in question number [1. b)] above. It should be about 10^{17} cm^{-3}. (**2 Marks**)

1. d) From the fundamental laws of thermodynamics, show that a quasi-static and adiabatic expansion of the universe demands that the black-body radiation of the universe satisfies the relation $R(t)T(t) =$ constant. (**4 Marks**)

1. e) From the relation derived in question number [1. d)] above, show that the energy density ($\tilde{u}(t)$) of the (relic) black-body radiation and the mass density ($\rho(t)$) of the matter (decoupled from radiation) in the expanding universe would satisfy the relation $\tilde{u}(t)\rho^{(-4/3)}(t) = \text{constant}$.
(**3 Marks**)

1. f) Using the informations obtained in question numbers [1. a)-d)] above, and by using the observational fact that, the present matter density of the universe is $\rho(t_{\text{present}}) \sim 10^{-30}$ gm cm^{-3}, determine the temperature of the (relic) black-body radiation in the present matter-dominated universe. Perhaps, it is roughly of the order $T(t_{\text{present}}) \sim (1-10)^{\text{o}}$ K, thus giving the approximate tempertaure of the cosmic microwave background radiation (CMBR).
(**3 Marks**)

2. a) A low mass (with mass M satisfying $M < M_{\odot} \sim 2 \times 10^{33}$ gm; $M_{\odot}$ is the solar mass) star, the nuclear fuel of which has already been exhausted, can still sustain itself against its self-gravity by contracting to such an extent that its inwardly directed (ie. directed from its surface towards its core) gravitational force may be balanced by the outwardly directed negative gradient of non-relativistic electron degenracy pressure in the stellar-interior. Show that such a star with radius R satisfies the relation $R \propto M^{(-1/3)}$. You may assume a highly simplifying constant density model of the star for the purpose of the this calculation.
(**5 marks**)

2. b) Next show that, for the negative gradient of electron degenracy pressure to balance the force of its self-gravity, the star must have a maximum possible mass limit (known as the Chandrasekhar mass limit). You do not have to calculate the numerical value of the above limit.
(**3 Marks**)

2. c) Briefly state, what might be the fate of the star if its mass exceeds the Chandrasekhar limit? Is there any more chance of its sustaining itself against self-gravity? Just state, if there is any possibility of having a second limiting mass like the Chandrasekhar mass, so that a star more massive than that limiting mass must collapse to a black-hole. In this question, you are not required to do any elaborate calculation to

estimate numerical values or to comment on the physics of black-holes.
(**2 Marks**)

3. a) Calculate the Fermi energy of a one-dimensional metal with one free electron per atom and an atomic spacing of 2.5×10^{-8} cm at $T = 0^{\circ}$ K, given that m_e (the rest-mass of electrons) $= 9 \times 10^{-28}$ gm and $\hbar$ (the reduced Planck's constant) $= 1.05 \times 10^{-27}$ erg s.
(**2 Marks**)

3. b) In order to escape from a metal the conduction electrons, whose motion within the metal is described by the free electron model, must overcome a potential barrier of height χ at the surface of the metal, ie. an electron will escape only if its component of velocity normal to the surface exceeds $(2\chi/m_e)^{1/2}$ with m_e as in question 3. a). What is the current density of electrons emitted from the surface of the metal when the temperature T is such that $\chi - \mu \gg k_B T$? Here, μ is the chemical potential of the electron gas and k_B is the Boltzmann constant.
(**8 Marks**)

4. a) Consider an ideal Boson gas composed of N particles in a volume V, and let N_{o} and N' be the number of particles in the lowest one-particle state (with momentum $\mathbf{p} = \mathbf{0}$) and in a higher state ($\mathbf{p} \neq \mathbf{0}$), respectively. Show that when the temperature falls below a critical temperature T_{c}, N_{o} suddenly becomes comparable with the total number N and that in this region the chemical potential μ is equal to zero. Here and in other problems on Bose condensation, you may like to use the Riemann zeta function: $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} dx}{(\exp(x) - 1)}$; $\zeta(\frac{3}{2}) = 2.61$, $\zeta(\frac{5}{2}) = 1.34$. Here, $\Gamma(s)$ is the Gamma function with $\Gamma(s+1) = s\Gamma(s) = s!$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
(**8 Marks**)

4. b) Calculate the critical temperature for ^{4}He with its mass density $\rho = 0.145 \text{ gm cm}^{-3}$. Given that (Boltzmann constant)$k_B = 1.38 \times 10^{-16}$ erg K^{-1} and the rest mass of a proton or neutron is $m_p \approx m_n = 1.67 \times 10^{-24}$ gm. The value of $\hbar$ is given in problem 3. a) above.
(**2 Marks**)

5. a) Starting from the grand partition function and the grand (thermodynamic) potential, calculate the energy and the entropy of an ideal boson system of N particles in the temperature region $0 \leq T \leq T_c$ with T_c

being the condensation temperature of the system.
(**6 Marks**)

5. b) Show that the energy and entropy are entirely due to non-condensate particles of the bose system mentioned above.
(**2 Marks**)

5. c) Show that the boson system in the above temperature region may be described in terms of a co-existence region of a first order phase transition, like water to ice phase transition, by clearly indicating the latent heat involved in that phase transition.
(**2 Marks**)

6. a) Show that the molar heat capacity of an ideal boson system at $T > T_c$ (T_c is the condensation temperature) is given by $\frac{C_{\mathrm{v}}}{N_{Av}k_B} = \frac{15}{4}\frac{\zeta_{5/2}(z)}{\zeta_{3/2}(z)} - \frac{9}{4}\frac{\zeta_{3/2}(z)}{\zeta_{1/2}(z)}$, in which $N_{Av} = 6 \times 10^{23}$ mol^{-1} is Avogadro's number and k_B is the Boltzmann constant. In the above equation, $\int_0^\infty \frac{x^s dx}{(\frac{1}{z}e^{\beta x} - 1)} = \frac{\Gamma(s+1)}{\beta^{s+1}}\zeta_{s+1}(z)$; $\beta = (1/(k_B T))$, $\Gamma(s)$ is the Gamma function and $\zeta_s(z)$ is the Bose-Einstein function with $\zeta_s(1) = \zeta(s)$. Here, $z = \exp(\beta\mu)$ is the fugacity of the non-interacting boson gas. Also, $z < 1$ for $T > T_c$, while $z = 1$ for $T \leq T_c$. In such problems, you might also like to use the generalized formula for derivatives of the Bose-Einstein functions, which is: $\left(\frac{\partial \zeta_n(z)}{\partial T}\right)_{V,N} = -\frac{3}{2T}\zeta_{n-1}(z)\left[\frac{\zeta_{3/2}(z)}{\zeta_{1/2}(z)}\right]$.
(**4 marks**)

6. b) Show that one of the two terms on the right hand side of the expression for molar heat capacity in problem [6. a)] is continuous as $T \leq T_c$, while the other term of that expression drops off as $T \to T_c$ from above. You might have to use the expansion $\zeta_n(\alpha) = \Gamma(1-n)\alpha^{n-1} + \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\zeta(n-l)\alpha^l$, $\alpha = -\ln z$ for the Bose-Einstein function with non-integer n at the singularity at $z = 1$ as $T \to T_c$ from above.
(**2 Marks**)

6. c) Use the expression for molar heat capacity in problem [6. a)] and the generalized formula for the derivative of Bose-Einstein function given in that problem to prove that $\left(\frac{\partial C_v}{\partial T}\right)_{V,N} > 0$ as $T \to T_c$ from below and

$\left(\frac{\partial C_v}{\partial T}\right)_{V,N} < 0$ as $T \to T_c$ from above.
(**2 Marks**).

6. d) Show that the molar heat capacity of the boson system reduces to that of a classical ideal gas in the limit $T \to \infty$.
(**2 Marks**)

7. a) Consider a gas of N non-interacting fermions in a fixed volume V and at a non-zero temperature $T = (\beta k_B)^{-1} \ll T_F$; T_F being the Fermi temperature of the system. At such a temperature, the chemical potential $(\mu(> 0))$ satisfies: $(\epsilon_F - \mu) \ll \epsilon_F$; $\epsilon_F = (k_B T_F)$ being the Fermi energy of the system. Show that, for a non-singular function $f(\epsilon)$, the Fermi-Dirac integral $\int_0^\infty \frac{f(\epsilon)d\epsilon}{\exp(\beta(\epsilon-\mu))+1}$ may be expanded about its value at $T = 0^o K$ to obtain: $I(\mu) \approx \int_0^{\epsilon_F} f(\epsilon)d\epsilon + (\mu - \epsilon_F) f(\epsilon_F) + \frac{\pi^2}{6}(k_B T)^2 f'(\epsilon_F)$. Here, $f'(\epsilon_F) = \frac{df(\epsilon)}{d\epsilon}|_{\epsilon_F}$.
(**6 Marks**).

7. b) From the results obtained in question number [7. a)] above, prove that the chemical potential of a free electron gas at a temperature T very small compared with the Fermi temperature T_F is given by $\mu(T) = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$.
(**4 Marks**).

8. Atoms in a solid vibrate about their respective equilibrium positions with small amplitudes. Debye approximated the normal vibrations with the elastic vibrations of an isotropic continuous body and assumed that the number of vibrational mode $g(\omega)d\omega$ having angular frequencies between ω and $\omega + d\omega$ is given by

$$\begin{aligned} g(\omega) &= \frac{V}{2\pi^2}\Big(\frac{1}{c_L^3} + \frac{2}{c_T^3}\Big)\omega^2 \equiv \frac{9N}{\omega_D^3}\omega^2 \qquad (\omega < \omega_D) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad (\omega > \omega_D) \end{aligned}$$

Here, c_L and c_T denote the velocities of longitudinal and transverse waves, respectively. The Debye frequency ω_D is determined by

$$\int_0^{\omega_D} g(\omega)d\omega = 3N \qquad (1)$$

where N is the number of atoms and hence $3N$ is the number of degrees of freedom. Calculate the specific heat of a solid at constant volume with this model. Examine its temperature dependence at high as well as low temperatures. Note that, the 'Debye function' is defined as: $\mathcal{D}(x_D) = \frac{3}{x_D^3}\int_0^{x_D} \frac{x^3 dx}{(\exp(x)-1)}$, that takes on an approximate form $\frac{(2\pi)^4}{80x_D^3}$ in the limit $x_D \gg 1$. In the limit $x_D \ll 1$, on the other hand, $\mathcal{D}(x_D) \approx 1$. (**10 Marks**).

9. a) Show that a two-dimensional non-relativistic ideal bose gas does not exhibit Bose-Einstein condensation.
(**4 Marks**)

9. b) Examine, whether a two-dimensional, ideal and ultra-relativistic bose gas exhibits Bose-Einstein condensation.
(**4 Marks**).

9. c) Is your conclusion in question number (9. c)) remains valid even for ultra-relativisitc, ideal three dimensional bose systems?
(**2 Marks**).