Five Year Integrated M.Sc. Examination, 2017

Semester-VI

Course: MT-3-6-2 (Old)

(Advanced Modern Algebra)

Time: Four Hours **Full Marks: 80**

Questions are of value as indicated in the margin.

Answer ***any eight*** questions.

1. a) What do you understand by a conjugate of an element in a group? 1
   b) Prove that in a finite group G, for each $a \in G$, the number of different conjugates of 'a' in G equals the number of distinct left cosets of the subgroup $C(a)$ in G. 5
   c) If G be a group of order p^2, where p is a prime, then show that G is abelian. 4

2. a) What is a p-group? Give an example. 2
   b) Show that every non-trivial finite p-group has non-trivial centre. 3
   c) State and prove Cauchy's theorem for finite groups. 1+4

3. a) State Sylow's first theorem.
   Using this or otherwise show that a group of order 3,15,000 has a subgroup of order 625. Can you say anything about a subgroup of order 30 of this group? 1+2+1
   b) What is Sylow p-subgroup? Find the number of Sylow 2-subgroups of S_3. 1+2
   c) Show that a Sylow p-subgroup of a finite group G is a normal subgroup of G if it is the only Sylow p-subgroup of G. 3

4. a) What are Nilpotent groups? Explain with an example. 2
   b) Show that every finite p-group is nilpotent. 6
   c) Define a solvable series in a group with an example. 2

5. a) Let X be a G-set. Then prove the following:
   i) The relation ρ on X defined by "$x\rho y$ if $gx = y$ for some $g \in G$" is an equivalence relation.
   ii) For each $x \in X$, $G_x = \{g \in G : gx = x\}$ is a subgroup of G. 5
   b) Let X be a left G-set, then defined orbit and stabilizer of an element $x \in X$. 2
   c) Let a group G act on a set X. Then show that $|orb(x)|$ is the index $[G : G_x]$ for every $x \in X$. 3

6. a) Let G be a group and H and K be two subgroups of G. Then prove that G is an internal direct product of H and K iff (i) $G = HK$, (ii) $ab = ba\ b \in K, \forall a \in H$ and (iii) $H \cap K = \{e\}$. 6
   b) Let A and B be two cyclic groups of order m and n respectively. Then prove that $A \times B$ is cyclic iff $\gcd(m,n) = 1$. 4

7. a) Prove that if an ideal S of a ring R with unity contains a unit of R, then $S = R$. 2
   b) Show that a division ring has no non-trivial proper ideals. 3
   c) Let R be a commutative ring with unity. Prove that a proper ideal M of R is maximal iff R/M is a field. 5

8. a) What is finite extension of a field? 1

P.T.O.

b) Prove that if L is a finite extension of a K and if K is a finite extension of F, then L is a finite extension of F. Also show that $[L:F]=[L:K]\cdot[K:F]$. 4+1

c) Let K be an extension of a field F. When an element $a \in k$ is said to be algebraic over F? Prove that an element $a \in k$ is algebraic over F if $F(a)$ is a finite extension of F. 1+3

9. a) Find the splitting field of $x^5+5x^3-2x^2-10$ over Q, the field of rational numbers and also find the order of such extension. 2+1

b) What do you understand by a simple extension of a field? Give an example of such type of extension. 2

c) If $k(u)$ is a simple extension of a field k, then prove the following:

(i) $k(u)=Q(k[u])$, (ii) $k(u)\simeq Q\left(\frac{k[x]}{p}\right)$, where p is a principal ideal in $k[x]$ consisting of all polynomials having u as a root. 5

10. a) Define a module over a ring R and give an example. 2

b) What are submodules of a R-module? 1

c) Show that any finite abelian group is the direct product of cyclic groups. 3

d) Let M be a R-module with additive identity O_m over a ring R with additive identity O_R, then prove that

i) $O_R \cdot x = r \cdot O_M = O_M \quad \forall r \in R$ and $x \in M$.

ii) $(-r)x=-(rx)=r(-x)\forall r \in R,\ x \in M$. 4

11. a) Define an Euclidean Domain and give an example. 2

b) Prove that every Euclidean Domain is a principal ideal domain. 3

c) Prove that in an integral domain, every prime is irreducible. 3

d) Show that if D is an integral domain, then $D[x]$ is an integral domain. 2