Five Year Integrated M.Sc. Examination, 2017

Semester - VI

Course: MT-3-6-1

Mathematics: Analysis-II

Time: Four Hours **Full Marks: 80**

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings.

Answer ***any eight*** questions.

1. a) Define neighbourhood of a subset of a topological space. 2

   b) Show that a subset 'A' of a topological space (X,τ) is open iff A is a neighbourhood of each of its points. 4

   c) Let U_1, U_2 be two neighbourhood of a point $x \in X$. Then show that

   i) If $U_1 \subseteq U_3$, then U_3 is a neighbourhood of x.

   ii) $U_1 \bigcap U_2$ is a neighbourhood of x. 2+2

2. a) Define open base of a topological space. 2

   b) Show that a collection B of open sets of a topological space (X, τ) forms an open base of τ iff $\forall \bigcup \in \tau$ and $\forall p \in U, \exists V_p \in B$ Subject to $p \in V_p \subseteq U$. 4

   c) For any two subsets A, B of a topological space (X, τ), show that $\overline{A \bigcup B} = \overline{A} \bigcup \overline{B}$. 2

   d) Give an example of a topological space which shows that $\overline{A \bigcap B} \neq \overline{A} \bigcap \overline{B}$. 2

3. a) Define subspace of a topological space. 2

   b) Let (X, τ) be a topological space and Y be a subset of X. Show that a point $p \in Y$ is a limiting point of a subset P of Y in (Y, τ_Y) iff p is a limiting point of P in (X, τ). 4

   c) Show that a mapping $f:(X, \tau) \to (Y, \tau^1)$ is open iff $\forall A \subseteq X, f(A^\circ) \subseteq [f(A)]^\circ$. 4

4. a) Define homeomorphism. 2

   b) Show that, for a bijective mapping $f:(X, \tau) \to (Y, \tau^1)$, the following conditions are equivalent: 4

   i) $f:(X, \tau) \to (Y, \tau')$ is open

   ii) $f:(X, \tau) \to (Y, \tau')$ is closed

   iii) $f^{-1}:(Y, \tau') \to (X, \tau')$ is continuous. 4

   c) Give an example of mapping which is neither open nor closed and continous. 4

5. a) Define T_0-space. 2

   b) Show that a topological space (X, τ) is T_0-space iff $\forall x, y \in X, x \neq y \Rightarrow \overline{\{x\}} \neq \overline{\{y\}}$. 4

   c) Show that co-finite topology is T_1-space but not a T_2-space. 4

6. a) Define regular space. 2

   b) Show that a topological space (X, τ) is regular iff $\forall x \in X$ and $\forall U \in \tau$ such that $x \in U, \exists V \in \tau$ such that $x \in V$ and $\overline{V} \subseteq U$. 5

   c) Give an example of a normal space. 3

P.T.O.

7. a) Let (X, τ) be a topological space. If $\exists$ an open base B of τ such that every cover of X by members of B has a finite sub cover, then show that (X, τ) compact. 4

   b) Show that a subset 'A' of the real number space with its natural topology is compact iff it is closed and bounded. 3

   c) Define locally compact space. Give an example of a locally compact space which is not compact. 3

8. a) Prove that a non-null subset E of (X, τ) is connected iff it can not be expressed as the union of two disjoint subsets A, B of E which are closed in (E, τ_E). 4

   b) Show that the union of a family of connected subsets of a topological space (X, τ), no two of which are separated is connected. 4

   c) Let $X = \{a,b,c,d\}$ and $\tau = \{\phi, \{a,b\}, \{c\} \cdot \{d\}, \{c,d\}, \{a \cdot b,c\}, \{a,b,d\}, X\}$. Then find the components of (X, τ). 2

9. a) Let (X, τ) and $(Y \cdot \tau^1)$ be any two topological spaces. Then show that the projection mappings are continuous and open. 5

   b) Show that $(X \times Y, \tau \times \tau^1)$ is connected iff (X, τ) and $(Y \cdot \tau^1)$ are connected. 5

10. a) State Uryshon's lemma. 2

    b) Show that a completely regular space is regular. 5

    c) Show that a T_4-space is completely regular space. 3

11. a) State and prove Tietz extension theorem. 2+6

    b) State Uryshon's metrization theorem. 2

12. a) Define n-dimensional manifold. 2

    b) Is the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ a chart? 3

    c) Show that the Euclidean plane is a differential manifold of dimension 2. 3

    d) Define immersion. 2