B.A./B.Sc. (Honours) Examination 2019

Semester – VI

Mathematics

Course: BMC-62

(Algebra-III)

Time: Three Hours Full Marks: 40

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

1. Answer ***any four*** questions. 4×5=20

a) Let H be a subgroup of a group G. If for each $a \in G$, there exists $b \in G$ such that $aH = Hb$, then prove that H is a normal subgroup of G.
Show that the quotient group $\mathbb{Q}/\mathbb{Z}$ is an infinite group but each of its elements is of finite order. 2+(2+1)

b) Let $f : G \to G'$ be a group homomorphism. If N is a normal subgroup of G' then show that $f^{-1}(N)$ is also normal in G.
If moreover, f is one-to-one then show that $0(a) = a(f(a))$ for every $a \in G$. 2+3

c) Find all homomorphisms from $(\mathbb{Z}_6, +)$ into $(\mathbb{Z}_8, +)$. Also show that there does not exist any isomorphism from the group $(\mathbb{R}, +)$ to the group $(\mathbb{R}^*, \cdot)$. 3+2

d) Define $f : \mathbb{C}^* \to \mathbb{R}^+$ by $f(z) = |z|$ for every $z \in \mathbb{C}^*$. Consider the equivalence relation ρ on $\mathbb{C}^*$ by : $z\rho w$ if $f(z) = f(w)$. Find the ρ-equivalence class of $2+3i$. Also find the inverse of $(2+3i)\rho$ in the group $\mathbb{C}^*/\rho$ of all ρ-equivalence classes of $\mathbb{C}^*$. Find a commonly known group to which the group $\mathbb{C}^*/\rho$ is isomorphic. 2+1+2

e) State and prove the second isomorphism theorem. If there exists an epimorphism from a finite group G onto the group $\mathbb{Z}_{15}$ then show that G has a normal subgroup of index 5. 3+2

f) Show that every commutative group of order 6 is cyclic. Find all subgroups of $\mathbb{Z}/24\mathbb{Z}$. 3+2

2. Answer ***any one*** question. 1×5=5

a) Show that the set $I = \left\{a + b\sqrt{3} \in \mathbb{Z}\left[\sqrt{3}\right] \,|\, a - b \text{ is an even integer}\right\}$ is an ideal in the ring $\mathbb{Z}\left[\sqrt{3}\right]$. List all distinct cosets of I in $\mathbb{Z}\left[\sqrt{3}\right]$. 2+3

b) Find all ring homomorphisms from $\mathbb{Z}_{12}$ into $\mathbb{Z}_{30}$. Show that the fields $Q\left(\sqrt{2}\right)$ and $Q\left(\sqrt{3}\right)$ are not isomorphic. 3+2

P.T.O.

3. Answer ***any three*** questions. $3\times5=15$

a) State and prove the rank-nullity theorem for linear transformations.
Find the rank of a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$, given that $\ker T$ is the x-axis in $\mathbb{R}^3$. 4+1

b) Show that a linear transformation $T:U \to V$ is a projection if and only if $T^2 = T$.
Define the projection of $\mathbb{R}^2$ on x-axis along $y = 2x$ line. 3+2

c) Define $T:P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ by $T(p(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$. Find $[T]_\alpha^\beta$ where $\alpha = \{1, x, x^2\}$ and $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.
Find $\ker T$ and nullity of T. 3+2

d) Prove that a matrix $A \in M_{n\times n}(F)$ is diagonalizable if and only if F^n has a basis of eigenvectors of A.
Is $A = \begin{pmatrix} 0 & 1 \\ -0 & 0 \end{pmatrix} \in M_{2\times2}(\mathbb{R})$ diagonalizable? 4+1

e) Let $\alpha = \{(1,-1,1), (0,1,1), (0,1,-1)\}$. Given that $T:\mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $[T]_\alpha^\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 3 & -2 \end{pmatrix}$. Find $[T(-2,1,0)]_\alpha$. Is T invertible?
Find $[T^{-1}]_\alpha^\alpha$. What is the nullity of T? 2+2+1

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