B.A./B.Sc. (Honours) Examination, 2017
Semester-VI
Mathematics (Honours)
Course: BMC-62
(Algebra-III)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

1. Answer ***any four*** questions: 4×5=20
   a) Let $f:G\to G$ be a group homomorphism. If $a\in G$ is of finite order, then show that $o(f(a))\,|\,o(a)$. Find all homomorphisms from $(\mathbb{Z}_6,+)$ into $(\mathbb{Z}_4,+)$. 2+3
   b) Show that $(\mathbb{Q},+)$ is not isomorphic to $(\mathbb{Q}^+,\cdot)$. Prove that there can not be any onto homomorphism from a group of order 9 onto a group of order 6. 3+2
   c) Find all subgroups of $\mathbb{Z}/21\mathbb{Z}$. Prove that every abelian group of order 6 is cyclic. 2+3
   d) Show that $GL_n(\mathbb{R})/SL_n(\mathbb{R})\simeq\mathbb{R}^*$. Let K be a normal subgroup of a finite group G. If G/K has an element of order n, then show that G has an element of order n. 3+2
   e) Let the identity permutation e in S_n be expressed as $e=\tau_1\tau_2\ldots\tau_r$ where τ_i are transpositions. Prove that r is an even integer. Express e as a product of transpositions in two ways one of which contains both (12) and (23). 4+1
   f) Prove that $|A_n|=\dfrac{\lfloor n}{2}$ [assume that $|S_n|=\lfloor n$]. Let $\beta\in S_n$ be such that $\beta^4=(2\ 1\ 4\ 3\ 5\ 6\ 7)$. Find β. 3+2

2. Answer ***any one*** question: 1×5=5
   a) State and prove the second isomorphism theorem for rings. Show that $\mathbb{Z}$ and $2\mathbb{Z}$ are not isomorphic as rings. 4+1
   b) Let F be a field. If there is a ring homomorphism from $\mathbb{Z}$ onto F, then show that $F\simeq\mathbb{Z}_p$ for some prime p. Prove that the fields $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic. 3+2

3. Answer ***any three*** questions: 3×5=15
   a) Define eigen values and eigen vectors of a square matrix $A\in M_n(F)$. Show that similar matrices have the same eigen values. Is the converse true? Justify. 1+2+2
   b) Find eigen values of
   $$A=\begin{pmatrix}1&-1&0\\1&2&1\\3&2&-2\end{pmatrix}.$$
   Also find the eigen space E_1 and dimension of E_1.
   If λ is an eigen value of an idempotent matrix then show that λ is either 0 or 1. 1+2+2

P.T.O.

c) Give an example of a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ having kernel the straight line $x+y+z=0=x+2y+3z$. Is it possible to obtain such an onto linear transformation? Justify. Give an example of a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $Ker\,T = lm\,T$. 2+1+2

d) State and prove the rank-nullity theorem. Find the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ that maps (1, 1) and (1, 0) into (1, 2) and (3, 4) respectively. 3+2

e) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x,y,z)=(2y+z,\,-x+4y+5z,\,x+z)$. Find the matrix representation of T relative to the basis $\beta=\{(1,1,1),(1,1,0),(0,1,1)\}$. Show that $R(T)=rank[T]_{\beta}$. 3+2