B.A./B.Sc. (Honours) Examination, 2017
Semester-IV
Mathematics (Core)
Course: BMC-41
(Analysis-IV)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and Symbols have their usual meanings.

Answer ***Question No. 1*** and ***Three*** from the rest.

1. Answer ***any ten*** questions: 10x1=10

a) Evaluate $\int_0^\infty x^4 e^{-x}dx$, if it converges.

b) Find the area in the first quadrant included between the parabola $x^2 = 4y$, the y-axis and the line $y = 25$.

c) Find the value of $\lambda(2) - \mu(2,1)$, where the tripple integral over the region bounded by the planes $2x + y + z = 4$, $x = 0$, $y = 0$ and $z = 0$ is given by $\int_0^2 \int_0^{\lambda(x)} \int_0^{\mu(x,y)} dz\,dy\,dx$.

d) Change the order of the integration in the integral $\int_0^{\pi/2} dx \int_0^{\cos x} x^2 dy$.

e) Find the condition for which the improper integral $\int_5^\infty x^\mu dx$ is convergent.

f) Find the value of the integration $\int_0^1 x^4 (1-x)^3\, dx$.

g) Find the value of $y(\pi)$, where $y(x) = \dfrac{d}{dx}\int_{\sin^2 x}^{2\sin x} e^{t^2} dt$.

h) Is the function $f(x) = \begin{cases} x\sin\dfrac{\pi}{x}; & x \in (0, 1] \\ 0; & x = 0 \end{cases}$,

a function of bounded variation on [0, 1].

i) Give an example of a bounded function for which lower and upper integral for a partition may not be equal. Justify your answer.

j) Use Second mean value theorem to show that $\left|\int_a^b \dfrac{\sin x}{x} dx\right| < \dfrac{4}{a}$.

k) For $K^2 > 1$, show that $\int_0^1 \dfrac{dx}{\sqrt{(1-x^2)(1-K^2x^2)}} \geq \dfrac{\pi}{6}$.

l) Construct a function $\phi(x)$ such that $\int_1^3 [x]dx = \phi(3) - \phi(1)$.

m) Define $f : [0, 1] \to \mathbb{R}$ by $f(x)\begin{cases} 1/x & \text{if } x > 0 \\ 1 & \text{if } x = 0. \end{cases}$ Examine if $(R)\int_0^1 f\,dx$ exists.

n) Give an example to show that the Riemann integral of a discontinuous function need not exist. Justify your answer.

P.T.O.

2. a) By using the transformation $x = r\cos\theta,\ y = r\sin\theta,$ show that
$\iint_E \frac{dxdy}{\left(1+x^2+y^2\right)^2} = \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{1}{2}\right)$, where E is the triangle with vertices at (0, 0), (2, 0) and $\left(1, \sqrt{3}\right)$. 4

b) If $f(x, \alpha)$ and $\frac{\partial f}{\partial \alpha}(x, \alpha)$ are continuous functions of x and α for $a \le x \le b,\ c \le \alpha \le d$, a, b being independent of α, then prove that $\frac{d}{d\alpha}\int_a^b f(x, \alpha)dx = \int_a^b \frac{\partial f}{\partial \alpha}(x, \alpha)dx.$

Using the above result, evaluate $\int_0^{\pi/2} \frac{dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)^2}.$ 2+4

3. a) Evaluate $\iint_R \sqrt{\left|x^2 - 2y\right|}\,dx\,dy$, where $R:[-2, 2;\ 0, 2]$. 5

b) Prove that $\iiint_V (lx+my+nz)^2\, dxdydz = \frac{4\pi}{15}\left(l^2+m^2+n^2\right)$, where V is taken throughout the sphere $x^2+y^2+z^2=1$. 5

4. a) Applying Dirichlet's test, show that $\int_0^\infty \cos\left(x^2\right)dx$ is convergent. 3

b) Express $\int_0^1 x^m\left(1-x^n\right)^p\, dx$ as a Beta function and $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$ as a Gamma function. 2

c) Let α be a monotonic increasing function on $[a,\ b]$ and is continuous at c where $a \le c \le b.$ Let f be a function be such that $f(c)=1$, and $f(x)=0$ for $x \ne c.$ Prove that f is Riemann-Stieltjes integrable and (RS) $\int_a^b f d\alpha = 0.$ 3

d) Show that $\int_\pi^{2\pi} \sin x\; d(\cos x) = -\frac{\pi}{2}.$ 2

5. a) A function f is bounded and integrable over $[a,\ b]$. If $\int_a^b [f(x)]^2\, dx = 0$, then prove that $f(x)=0$ at all points of continuity of f. 4

b) If f is Riemann integrable over $[a,\ b]$ and continuous at $c,\ a<c<b,$ then prove that $\int_a^x f(t)dt;\ a \le x \le b,$ is differentiable at $x=c.$ 4

c) Show that the function $f:[0, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} x^2\cos\frac{1}{x} & ;\ \text{if } x \ne 0 \\ 0 & ;\ \text{if } x = 0, \end{cases}$

is a function of bounded variation on [0, 1]. 2

6. a) Show that if $f(x)$ be monotone on $[a,\ b]$, $\int_a^b f dx$ lies in between $f(a)(b-a)$ and $f(b)(b-a).$ 3

b) Let $f(x)$ be Riemann integrable and nonnegative on $[a,\ b]$. If $f(x)$ be continuous at c on $[a,\ b]$ and $f(c)>0,$ then prove that $\int_a^b f(x)dx > 0.$ 3

c) Find $\lim_{x\to 0} \frac{\int_0^x \sin t\, dt}{x^2}$, it it exists. 2

d) Let $f(x) = \begin{cases} x & \text{if } x \ne \frac{1}{n} \\ 1 & \text{if } x = \frac{1}{n};\ \text{for } n = 1, 2, \ldots . \end{cases}$

Is f Riemann integrable on [0, 1]? Give reasons. 2

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