B.A./B.Sc. (Honours) Examination, 2017
Semester-II
Mathematics (Core)
Course: BMC-22
(Vector Analysis & Number Theory)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Symbols have their usual meanings.

Unit-I (Marks-24)

Vector Analysis

Answer *any* ***four*** questions.

1. If $\vec{a}, \vec{b}, \vec{c}$ be any three noncoplanar vectors and $\vec{A}_i = x_i\vec{a} + y_i\vec{b} + z_i\vec{c}\,(i = 1, 2, 3)$ prove that

$$\left[\vec{A}_1\ \vec{A}_2\ \vec{A}_3\right] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \left[\vec{a}\ \vec{b}\ \vec{c}\right].$$

Hence or otherwise find $\vec{a}, \vec{b}, \vec{c}$ in terms of $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}, \vec{b}' = \dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}, \vec{c}' = \dfrac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}.$ 3+3

2. Define tangent vector, principal normal vector and binormal vector for a curve $\vec{r} = \vec{r}(t)\left(\equiv x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}\right)$. Find the equation for the (a) tangent, (b) principal normal and (c) binormal to the curve $x(t) = 3\cos t,\ y(t) = 3\sin t,\ z(t) = 4t$ at the point $t = \pi.$ 1.5+4.5

3. Obtain the value of $\vec{\nabla}\times\left(\vec{\nabla}\phi\right)$ and $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)$ for any smooth scalar field $\phi(x, y, z)$ and vector field $\vec{A}(x, y, z)$. Eliminate the curl operator $\left(\vec{\nabla}\times\right)$ from $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)$. Find $\nabla^2\left(\dfrac{1}{r}\right)$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$ 2+2+2

4. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(\vec{r}) = (2x + y)\hat{i} + (37 - x)\hat{j}$ and the curve C in the xy-plane consists of line segments joining (0, 0) to (2, 0) and then to (2, 2).
Also find the surface integral $\iint_S \vec{r}\cdot\hat{n}\,ds$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9,\ y = 0,\ y = 8.$ 3+3

5. State Stoke's theorem. Show that Green's theorem is a special case of Stoke's theorem.
If $\vec{H} = \vec{\nabla}\times\vec{A},$ for any smooth vector field $\vec{A}(x, y, z),$ find the value of $\iint_S \vec{H}\cdot d\vec{S}$ for any smooth closed surface S. 4+2

6. State and prove Green's first and second identities. 2+4

P.T.O.

Unit-II (Marks-16)

Number Theory

Answer Question No. 6 and *any* ***one*** from the rest.

6. Answer all the questions: 6×1=6

(i) What is the unit digit in $7^{7^{7^{\cdots 7}}}$

(ii) Use theory of congruence to verify $73 \mid 2^{36} - 1$.

(iii) Find the missing digit in the number 23104__791 if it is divisible by 7.

(iv) Let *a* and *b* be two integers not both zero. Consider the set $S = \{ax + by > 0 : x, y \in \mathbb{Z}\}$. Show that $\gcd(a, b)$ is the least member of *S*.

(v) Let *p* be a prime such that $n < p < 2n$, where *n* is a positive integer. Show that $2n_{c_n} \equiv 0 (\mathrm{mod}\, p)$.

(vi) Let a be a positive integer not divisible by 5. Use Euler's theorem to show that last digits of *a* and a^5 are same.

7. a) How many integers in the sequence 11, 111, 1111, 11111, ... are perfect squares. 2

b) If *a* be an arbitrary integer then show that $360 \mid a^2 (a^2 - 1)(a^2 - 4)$. 3

c) Let $n \geq 1$ and *a, b* be two positive integers. If $\gcd(a^n, b^n) = a^n$ then show that $\gcd(a, b) = a$. 2

d) Verify that any integer *n* can be expressed as $n = p^k m$, where $k \geq 0$, *p* is prime and $\gcd(m, p) = 1$. 3

8. a) Use the theory of congruence to show that any set of *n* consecutive integers is divisible by *n*. 3

b) For any $n \geq 1$, show that there exists a prime with at least *n* of its digits equal to 0. 3

c) If *m* and *n* are relatively prime positive integers, then show that

$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 (\mathrm{mod}\, mn).$$ 2

d) Use Fermat's theorem to prove that, if *p* is an odd prime, then

$$1^p + 2^p + 3^p + \dots + (p-1)^p \equiv 0 (\mathrm{mod}\, p).$$ 2