M.A./M.Sc. Examination, 2017

Semester - II

Mathematics

Paper: MMC-24

(Partial Differential Equations)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

(Notations have their usual meaning unless otherwise stated)

Answer ***any four*** questions

1. a) Apply $\sqrt{u} = v$ and $v(x, y) = f(x) + g(y)$ to solve the PDE $x^4 u_x^2 + y^2 u_y^2 = 4u$. Hence, find the singular integral of the PDE, if it exists. 4+1

   b) Reduce $u_{xx} + y^2 u_{yy} = y \quad (y > 0)$ to canonical form. 5

2. a) Find a surface passing through the parabolas $u = 0,\ y^2 = 4ax$ and $u = 1,\ y^2 = -4ax$ satisfying $xu_{xx} + 2u_x = 0$. 3

   b) Determine the type of the PDE

   $$u_{xx} + 2u_{yz} + \cos x\, u_z - e^{y^2} u = \cosh z,\ u = u(x, y, z).$$ 2

   c) Solve the following PDE by using Monge's method:

   $$x^2 u_{xx} - y^2 u_{yy} - xu_x + yu_y = xy.$$ 5

3. a) Find the expression of the Monge cone of the PDE

   $$x^2 u_x^2 + y^2 u_y^2 = u^2 \text{ at } (1, 2, 3).$$ 2

   b) Determine the characteristic strips of the PDE $u = u_x^2 - 3u_y^2$ and obtain the integral surface which passes through the curve $x = s,\ y = 0,\ u = s^2,\ s$ is the parameter. 4

   c) Obtain the solution of the free vibrations of a semi-infinite string governed by

   PDE: $u_{tt} = c^2 u_{xx},\ 0 < x < \infty,\ t > 0$

   IC's: $u(x, 0) = f(x)$

   $u_t(x, 0) = g(x)$

   $u, u_x \to 0$ as $x \to \infty$. 4

4. a) What do you mean by Dirichlet-type and Neumann-type boundary conditions for a PDE? Are the solutions corresponding to above types boundary conditions unique? Explain. 1+2

   b) Separate $u_x + 2u_{tx} - 10u_{tt} = 0$,

   $u(0, t) = 0,\ u_x(l, t) = 0$

   into an equation of x and an equation of t. 2

   c) Obtain the solution by separation of variables method of the following IBVP for the Laplace equation:

   $$u_{xx} + u_{yy} = 0 \text{ for } 0 < x < \pi \text{ and } 0 < y < \pi$$

   where $u(x, 0) = x^2,\ u(x, \pi) = 0,$

   $u_x = 0$ at $(0, y)$ and (π, y). 5

P.T.O.

5. a) Use Laplace transform to solve:

PDE: $u_{tt} = u_{xx}$, $0 < x < 2017,\ t > 0$

BC's: $u(0,\ t) = u(2017,\ t) = 0,\ t > 0$ 5

IC's: $u(x,\ 0) = \sin(\pi x),\ u_t\big]_{(x,\ 0)} = -\sin(\pi x),\ 0 < x < 2017.$

b) Use the method of integral transform to show that

$$u(x,\ t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) e^{-\frac{(x-v)^2}{4t}}\, dv$$

is the solution of the heat equation

$u_t = u_{xx},\ -\infty < x < \infty,\ t > 0$ 5

with $u = f(x)$ when $t = 0$, $u(x,\ t)$ is bounded.

6. a) Write the Hadamard's conditions for a well-posed PDE. 2

b) Solve the following interior Dirichlet problem for a circle:

PDE: $\nabla^2 u = 0,\ 0 \le r < a,\ 0 \le \theta < 2\pi$

BC: $u(a,\ \theta) = f(\theta),\ 0 \le \theta < 2\pi$

where $f(\theta)$ is a continuous function on $\partial\mathbb{R}$.

Hence deduce Poisson's integral formula of the form

$$u(r,\ \theta) = \frac{1}{2\pi} \int_{\varphi=0}^{2\pi} \frac{(a^2 - r^2) f(\theta)}{a^2 - 2ar\cos(\varphi - \theta) + r^2}\, d\varphi.$$ 6+2