M.A./M.Sc. Examination, 2017

Semester - II

Mathematics

Paper: MMC-23

(Algebra-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings

Answer ***any four*** questions

1. a) Let R be an integral domain. Then show that the units of $R[x]$ are contained in R. 3

   b) Show that $[4]x^2+[2]x+[4]$ is a zero divisor and $[4]x+[3]$ unit in $\mathbb{Z}_8[x]$. 2

   c) Let $(E, +, \cdot, v)$ be a Euclidean domain. How many Euclidean valuation can you construct from the Euclidean valuation v? Justify you answer. 2

   d) Let R be a commutative ring with unity 1. If $R[x]$ is a principal ideal domain, show that $R[x]$ is a Euclidean domain. 3

2. a) In the ring $\mathbb{Z}\left[i\sqrt{5}\right]$, show that $\gcd\left(6\left(1-i\sqrt{5}\right)\right), 3\left(3+i\sqrt{5}\right)\left(1-i\sqrt{5}\right)$ does not exist. 3

   b) Let R be a principal ideal ring. Show that every irreducible element of R is prime. 3

   c) Let R be a field. Then show that $R[x]/\langle p(x)\rangle$ is an integral domain if and only if $\langle p(x)\rangle$ is maximal in $R[x]$. 4

3. a) Show that every Euclidean domain is a factorization domain. 3

   b) Show that every irreducible polynomial in $\mathbb{Z}[x]$ is primitive. 2

   c) Let F be a field and $f(x)\in F[x]$ be a polynomial of degree 2 or 3. If $f(x)$ has no root in F, show that f is irreducible over F. 3

   d) Give an example of a polynomial having no roots in a field $\mathbb{F}$ reducible in $\mathbb{F}[x]$. 2

4. a) Let T be a linear operator on $P_2(\mathbb{R})$ defined by

$$T(f(x)) = f(1) + f'(0)x + \left(f''(0)+f''(0)\right)x^2.$$

   Test the diagonalizability of T. 4

   b) Find the characteristic and minimal polynomial of the matrix. 3

$$\begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

   c) Show that a linear operator T on a finite dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigenspaces of T. 3

5. a) Let T be a linear operator on a finite dimensional vectorspace V such that the characteristic polynomial of T splits. Let λ be an eigen value of T having geometric multiplicity m. Then prove that $\dim G_\lambda = m$. 4

P.T.O.

b) For the linear operator T on $M_{2\times2}(\mathbb{R})$, defined by $T(A) = 2A + A^t$ for all $A \in M_{2\times2}(\mathbb{R})$, find a basis β of $M_{2\times2}(\mathbb{R})$ which is union of disjoint cycles of generalized eigen vectors of T. 4

c) Let T be a linear operator on a finite dimensional vector space V whose characteristic polynomial splits. Then show that $\beta \cap G_\lambda$ is a basis of G_λ for each Jordan cannonical basis β of V. 2

6. a) Let V be a finite dimensional inner product space over F and let $g: V \to F$ be a linear transformation. Then show that there is a unique vector $y \in V$ s.t $g(x) = \langle x, y \rangle$ for all $x \in V$. 5

b) Let T be a linear operator on an inner product space V. Show that $\|T(x)\| = \|x\|$ for all $x \in V$ if and only if $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$. 5