# M.A./M.Sc. Examination 2017

**Semester - II**

**Mathematics**

**Course: MMC-21**

**( Functional Analysis )**

Time: Three Hours | Full Marks: 40

Questions are of value as indicated in the margin.

Notations and symbols have their usual meanings.

*Answer any four questions.*

1. a) When is a set a metric space $(X,d)$ called (i) bounded and (ii) totally bounded. Show that a totally bounded set is bounded but converse is not true. 6

   b) Show that if a discrete metric space $X$ is compact then $X$ contains finitely many points. 2

   c) Prove that every compact set in a metric space is closed and bounded. 2

2. a) When are two norms on a linear space said to be equivalent? Prove that any two norms defined on a finite dimensional linear space are equivalent. 3

   b) Show that $C[0,1]$ is a Banach space *w.r.t.* a norm to be defined by you. 3

   c) State Riesz Lemma and hence show that if a closed unit ball in a normed linear space $X$ is compact then $X$ is finite dimensions. 4

3. a) When is a linear operator $T$ over a normed linear space said to be bounded? Prove that every linear operator defined on a normed linear space is continuous iff it is bounded. 1+4

   b) Give an example with justifications of a linear operator over a normed linear space which is not bounded. 3

   c) Examine if the function $\|\cdot\|$ defined on $\mathbb{R}^2$ by $\|x\| = \left(|x_1|^{1/2} + |x_2|^{1/2}\right)^2$, $x = (x_1, x_2) \in \mathbb{R}^2$ is a norm. 2

4. a) State and prove closed Graph Theorem. 7

   b) Let $(a_1, a_2, a_3, ...) \in \ell_2$ and define a functional $f$ on $\ell_2$ by $f(x) = \sum_{i=1}^{\infty} a_i x_i$, $x = (x_1, x_2, ...) \in \ell_2$. Show that $f$ is a bounded linear functional such that $\|f\| = \left(\sum_{i=1}^{\infty} |a_i^2|\right)^{1/2}$. 3

5. a) In an inner product space $(X,<>)$, prove that $|\langle x, y\rangle|^2 \le \|x\|\|y\| \; \forall x, y \in X$, where $\|x\|^2 = \langle x, x\rangle$ and hence show that $X$ is a normed linear space *w.r.t.* the norm induced by the inner product. 5

b) Prove that $\mathbb{R}^n$ is an inner product space *w.r.t.* an inner product to be defined by you. 3

c) Show that $C[a,b]$ *w.r.t.* sup norm is not a Hilbert space. 2

6. a) Define an orthonormal set in an inner product space. Prove that any orthonormal set of vectors in an inner product space is linearly independent. 3

b) Show that closure of a convex set in a normed linear space is convex. 2

c) Define weak convergence and strong convergence of a sequence of elements in a normed linear space. Show that in a finite dimensional normed linear space $X$ weak convergence and strong convergence of a sequence of elements in $X$ are equivalent. 5

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