M.A./M.Sc. Examination, 2017
Semester-IV
Mathematics
Course: MMO-41 (P5)
(Rings and Modules-II)

Time: Three Hours **Full Marks: 40**

Questions are of value as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions 4×10=40

1. a) Prove that a right *R*-module *M* is Noetherian iff every submodule of *M* is finitely generated. 4

   b) Let $O \longrightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow O$ be a short exact sequence of right *R*-modules and *R*-morphisms. Prove that *M* is Artinian iff both M_1 and M_2 are Artinian. 3

   c) Let *R* be a right Artinian ring and $a, b \in R$ be such that $ab = 1$. Then show that $ba = 1$. 3

2. a) Define injective right *R*-modules. Prove that a right *R*-module *M* is injective iff for every right *R*-module *N*, and submodule N' of *N*, every *R*-morphism $f : N' \to M$ can be extended to an *R*-morphism $g : N \to M$. 4

   b) Show that external direct sum $\bigoplus_{\alpha \in \Delta} M_\alpha$ of *R*-divisible modules is *R*-divisible. 3

   c) Prove that every *R*-module can be embedded into an injective right *R*-module. 3

3. a) Prove that every free module is projective. Is the converse true? Justify. 3

   b) Prove that a right *R*-module *M* is projective iff *M* is isomorphic to a direct summand of a free *R*-module. 4

   c) Let *M* be a projective right *R*-module. Prove that the functor $Hom(M, -) : Mod_R \to Ab$ is an exact functor. 3

4. a) Prove that $Rad\left(\bigoplus_{\alpha \in \Delta} M_\alpha\right) = \bigoplus_{\alpha \in \Delta} Rad\, M_\alpha$. 4

   b) Show that $J(R) = \{a \in R \,|\, 1 + \text{ras is a unit for every } r, s \in R\}$. 3

   c) Show that $M_n(D)$ is *J*-semisimple, where *D* is a division ring. 3

5. a) Prove that a ring *R* is semiprime iff it is a subdirect product of prime rings. 4

   b) Prove that every ring is a subdirect product of subdirectly irreducible rings. 3

   c) Show that every Boolean ring is a subdirect product of copies of the field $\mathbb{Z}_2$. 3

6. a) Let *R* be a right semisimple ring. Prove that *R* is a direct sum of finitely many minimal right ideals. Hence or otherwise show that every right semisimple ring is both right Artinian and right Noetherian. 4

   b) Prove that a ring *R* is right semisimple iff every right *R*-module is semisimple. 3

   c) Let $M = \bigoplus_{i=1}^{n} M_i$ where each M_i is a simple right *R*-module and $M_i \simeq L$ for some simple right *R*-module *L*. Prove that $End_R(M) \simeq M_n(End_R(L))$. 3

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